Unconditionally Stable Transport Solver for Polymer.

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Plan

1. Polymer model equations.
2. Discretization and Splitting of the residual equations.
3. Unconditional Stability for the transport solver.

Derivation of the Todd-Longstaff polymer model

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- Conservation of mass for the polymer phase and the pure water phase:

$$\frac{\partial}{\partial t}(\phi S_{wp}) + \nabla \cdot \left(\frac{k_{wp}(S_{wp})}{\mu_{wp}} \boldsymbol{K} \nabla P \right) = 0,$$
$$\frac{\partial}{\partial t}(\phi S_{ww}) + \nabla \cdot \left(\frac{k_{ww}(S_{ww})}{\mu_{ww}} \boldsymbol{K} \nabla P \right) = 0.$$

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- We sum up the two equations of mass conservation ...

$$\frac{\partial}{\partial t}(\phi S_{wp}) + \nabla \cdot \left(\frac{k_{wp}(S_{wp})}{\mu_{wp}} \boldsymbol{K} \nabla P \right) = 0$$

$$\frac{\partial}{\partial t}(\phi S_{ww}) + \nabla \cdot \left(\frac{k_{ww}(S_{ww})}{\mu_{ww}} \boldsymbol{K} \nabla P \right) = 0$$

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- ... and obtain

$$\frac{\partial}{\partial t}(\phi S_w) + \nabla \cdot \left(\left(\frac{c}{c_{\max}} \frac{1}{\mu_{wp}} + (1 - \frac{c}{c_{\max}}) \frac{1}{\mu_{ww}} \right) k_w(S_w) \boldsymbol{K} \nabla P \right) = 0$$

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$$\frac{\partial}{\partial t}(\phi S_w) + \nabla \cdot \left(\left(\qquad \frac{1}{\mu_{w,\mathrm{eff}}} \qquad \right) k_w(S_w) \boldsymbol{K} \nabla P \right) = 0,$$

The equation of mass conservation for water.

The mixing parameter ω

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- If $\mu_{ww}(c) = \mu_{wp}(c)$: The two phases are identical. No interaction, polymer is simply transported.
- Intermediate cases are defined by introducing a parameter ω

$$\begin{cases} \mu_{ww}(c) = \mu_m(c)^\omega \mu_w^{1-\omega} \\ \mu_{wp}(c) = \mu_m(c)^\omega \mu_p^{1-\omega} \end{cases}$$

where μ_m denotes the concentration of the fully mixed solution.

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- We look at the one dimensional problem: 2x2 system of hyperbolic conservation laws, with given total flux.

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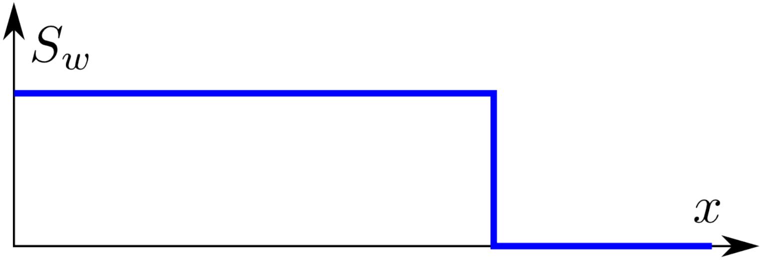

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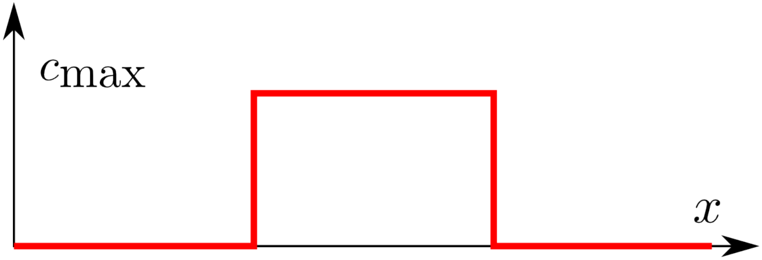

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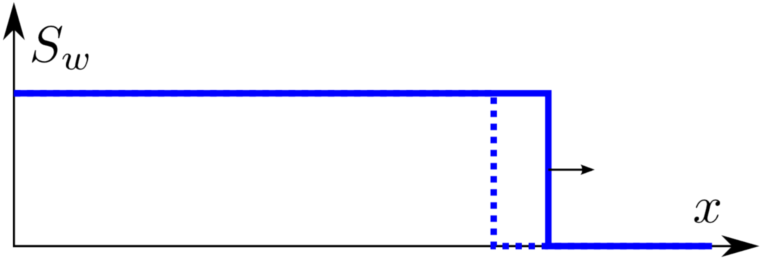

For all ω, the water front propagates at same speed.

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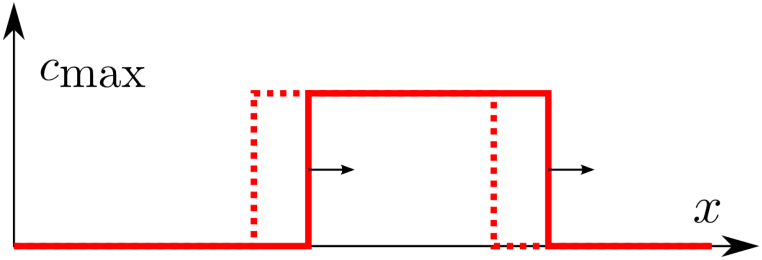

Case $\omega = 1$, fully mixed.

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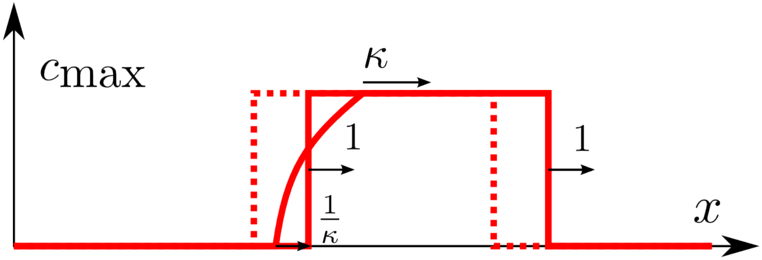

Case $\omega < 1$, rarefaction wave at the tail.

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- Adsorption.
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- Dead pore space. There is a problem with the model in eclipse. We use an alternative one.
- Finally, mass conservation equations:

$$\frac{\partial}{\partial t}(b_w \phi S_w) + \nabla \cdot (b_w \vec{v}_w) = 0,$$

$$\frac{\partial}{\partial t}(b_o \phi S_o) + \nabla \cdot (b_o \vec{v}_o) = 0,$$

$$\frac{\partial}{\partial t}(b_w \phi S_w c) + \frac{\partial}{\partial t}\left((1 - \phi_{\text{ref}})\hat{c}^a\right) + \nabla \cdot (b_w c \vec{v}_{wp}) = 0,$$

with

$$\vec{v}_w = -\frac{k_{rw}}{\mu_{w,\text{eff}}(c) R_k(c^a)} \boldsymbol{K}(\nabla p_w - \rho_w g \nabla z),$$

$$\vec{v}_{wp} = -\frac{k_{rwp}}{\mu_{p,\text{eff}}(c) R_k(c^a)} \boldsymbol{K}(\nabla p_w - \rho_w g \nabla z) = m(c)\vec{v}_w.$$

Discretization and Splitting

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- Time discretization: Implicit.

Water and oil residuals (only advection part)

Water and oil residual for the cell i:

$$\begin{aligned} R_{\alpha,i}(S^{n+1}, c^{n+1}) =& b_i^{n+1}\phi_i^{n+1}S_i^{n+1} - b_i^n\phi_i^n S_i^n \\ &+ \frac{\Delta t}{V_i} \sum_{\{j \mid v_{i,j}^{n+1}<0\}} f_\alpha(S_j^{n+1}, c_j^{n+1}) b_{ij}^{n+1} v_{ij}^{n+1} \\ &+ \frac{\Delta t}{V_i} f_\alpha(S_i^{n+1}, c_i^{n+1}) \sum_{\{j \mid v_{i,j}^{n+1}>0\}} b_{ij}^{n+1} v_{ij}^{n+1}, \end{aligned}$$

for $\alpha = \{w, o\}$

Polymer residual (only advection part)

Polymer residual for the cell i:

$$
\begin{aligned}
R_{c,i}(S^{n+1}, c^{n+1}) = b_i^{n+1}\phi_i^{n+1}S_i^{n+1}c_i^{n+1} + \hat{c}^a(c_i^{n+1})(1-\phi_{\mathsf{ref},i}) \\
&- \left(b_i^n\phi_i^n S_i^n c_i^n + \hat{c}^a(c_i^n)(1-\phi_{\mathsf{ref},i})\right) \\
&+ \frac{\Delta t}{V_i}\sum_{\{j|v_{i,j}^{n+1}<0\}} m(c_j^{n+1})c_j^{n+1}f_w(S_j^{n+1}, c_j^{n+1})b_{ij}^{n+1}v_{ij}^{n+1} \\
&+ m(c_i^{n+1})c_i^{n+1}f_w(S_i^{n+1}, c_i^{n+1})\frac{\Delta t}{V_i}\sum_{\{j|v_{i,j}^{n+1}>0\}} b_{ij}^{n+1}v_{ij}^{n+1} \\
= 0.
\end{aligned}
$$

Preconditioning versus splitting

- We can solve the fully coupled system of equations. Everything is let to the Newton solver. The physics is lost (even if it may come back via preconditioners).

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- We can solve the fully coupled system of equations. Everything is let to the Newton solver. The physics is lost (even if it may come back via preconditioners).
- Alternatively, we split the equations. Each splitting part corresponds to a well-identified physical phenomenon. We regain control on the computation process.

The pressure equation

- Assume no polymer, incompressible fluids. Then, mass conservation for each phase is given by

$$\frac{\partial(\phi S_\alpha)}{\partial t} + \nabla .\left(\lambda_\alpha(S_\alpha)\boldsymbol{K}\nabla P\right) = 0$$

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- After summation, it yields

$$\begin{aligned}\phi\frac{\partial}{\partial t}(\sum_\alpha S_\alpha - 1) = &-\frac{\partial\phi}{\partial t}(\sum_\alpha S_\alpha - 1)\\ &-\left(\frac{\partial\phi}{\partial t} - \nabla.(\sum_\alpha(\lambda_\alpha(S_\alpha))\boldsymbol{K}\nabla P)\right)\end{aligned}$$

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- The pressure enforces volume preservation.

Single cell problem

- Discrete pressure equation

$$\frac{1}{b_{w,i}} R_{w,i}(S^n, c^n) + \frac{1}{b_{o,i}} R_{o,i}(S^n, c^n) = 0.$$

Single cell problem

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$$\frac{1}{b_{w,i}} R_{w,i}(S^n, c^n) + \frac{1}{b_{o,i}} R_{o,i}(S^n, c^n) = 0.$$

- Transport equation (advection part). The single cell problem for the cell i is given by

$$\begin{aligned} R_{w,i}(S_i^{n+1}, c_i^{n+1}) = {} & b_i^{n+1}\phi_i^{n+1} S_i^{n+1} - b_i^n \phi_i^n S_i^n \\ & + \frac{\Delta t}{V_i} \sum_{\{j \mid v_{i,j}^{n+1} < 0\}} f_w(S_j^{n+1}, c_j^{n+1}) b_{ij}^{n+1} v_{ij}^{n+1} \\ & + \frac{\Delta t}{V_i} f_w(S_i^{n+1}, c_i^{n+1}) \sum_{\{j \mid v_{i,j}^{n+1} > 0\}} b_{ij}^{n+1} v_{ij}^{n+1} \\ = {} & 0 \end{aligned}$$

Uniform stability

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- Definition: We say that the transport solver is **uniformly stable** if there exists a unique solution to the single cell problem for any time step Δt.

Simple case for advection

- We demonstrate the proof of unconditional stability in the simplest case. We want to solve

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$$R_w(0,c) = -\phi_i^n S_i^n + \frac{\Delta t}{V_i} \sum_{\{j \mid v_{i,j}^{n+1} < 0\}} f_w(S_j^{n+1}, c_j^{n+1}) v_{i,j}^{n+1} \leq 0.$$

$$\begin{aligned} R_w(1,c) &= R_{w,i}(1,c) - (R_{w,i}(S^n,c^n) + R_{o,i}(S^n,c^n)) \\ &= \phi_i^n S_{o,i}^n - \frac{\Delta t}{V_i} \sum_{\{j \mid v_{i,j}^{n+1} < 0\}} (1 - f_w(S_j^{n+1}, c_j^{n+1})) v_{i,j}^{n+1} \\ &\geq 0 \end{aligned}$$

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- We check that $R_w(0,c) \leq 0$, $R_w(1,c) \geq 0$ and $\frac{\partial R_w}{\partial S} > 0$.
- We look at function $c \mapsto R_c(S(c),c)$ and check that:

$$\begin{cases} R_c(S(0),0) \leq 0, \\ R_c(S(c_{\max}),c_{\max}) \geq 0, \\ \dfrac{d}{dc}(R_c(S(c),c)) > 0. \end{cases}$$

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- We use essentially that

$$\frac{\partial f_w}{\partial S} \geq 0 \quad \text{and} \quad \frac{\partial f_w}{\partial c} \leq 0$$

General case for advection

We can prove that the single cell transport solver is uniformly stable for

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Segregation case - without polymer

- The segregation equation is given by

$$\frac{\partial}{\partial t}(\phi S_w) + \nabla \cdot \left(\frac{\lambda_w(S_w)\lambda_o(S_o)}{\lambda_w(S_w) + \lambda_o(S_o)}(\rho_o - \rho_w) g \boldsymbol{K} \nabla z \right) = 0$$

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- The residual for water mass conservation is given by

$$R_w(S) = \phi_i(S_i - S_i^*) + \frac{g\Delta t}{V_i} \Big(F(S_i, S_{i+1}) T_{i,i+1}(z_{i+1} - z_i) - F(S_{i-1}, S_i) T_{i-1,i}(z_i - z_{i-1}) \Big) = 0$$

where $F(S_u, S_l)$ approximates the flux for segregation,

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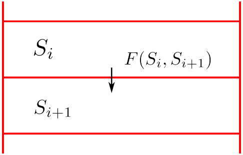

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- Upwind phase mobility:

$$F(S_u, S_l) = \frac{\lambda_w(S_u)\lambda_o(1 - S_l)}{\lambda_w(S_u) + \lambda_o(1 - S_l)}.$$

Segregation case - with polymer

- Let $\boldsymbol{u} = (S, c)$. The residuals take the form

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and

$$\phi_i(S_i c_i - S_i^* c_i^*) + \frac{g\Delta t}{V_i}\Big(G(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})T_{i,i+1}(z_{i+1} - z_i) - G(\boldsymbol{u}_{i-1}, \boldsymbol{u}_i)T_{i-1,i}(z_i - z_{i-1})\Big) = 0$$

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and

$$\phi_i(S_i c_i - S_i^* c_i^*) + \frac{g\Delta t}{V_i}\Big(G(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})T_{i,i+1}(z_{i+1} - z_i) - G(\boldsymbol{u}_{i-1}, \boldsymbol{u}_i)T_{i-1,i}(z_i - z_{i-1})\Big) = 0$$

- We can prove that the fluxes given by

$$F(\boldsymbol{u}_u, \boldsymbol{u}_l) = \frac{\lambda_w(S_u, c_u)\lambda_o(1 - S_l)}{\lambda_w(S_u, c_u) + \lambda_o(1 - S_l)},$$

$$G(\boldsymbol{u}_u, \boldsymbol{u}_l) = m(c_u)c_u\frac{\lambda_w(S_u, c_l)\lambda_o(1 - S_l)}{\lambda_w(S_u, c_l) + \lambda_o(1 - S_l)}.$$

yield stability.