Unconditionally Stable Transport Solver for Polymer.

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Plan

1. Polymer model equations.
2. Discretization and Splitting of the residual equations.
3. Unconditional Stability for the transport solver.
We use an immiscible model for polymer.
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Polymer phase ($S_{wp}$) and pure water phase ($S_{ww}$):

\[
S_{wp} = \frac{c}{c_{\text{max}}} S_w, \quad S_{ww} = (1 - \frac{c}{c_{\text{max}}}) S_w.
\]
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Polymer phase \( (S_{wp}) \) and pure water phase \( (S_{ww}) \):

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\]

Conservation of mass for the polymer phase and the pure water phase:

\[
\frac{\partial}{\partial t} (\phi S_{wp}) + \nabla \cdot \left( \frac{k_{wp}(S_{wp})}{\mu_{wp}} K \nabla P \right) = 0,
\]

\[
\frac{\partial}{\partial t} (\phi S_{ww}) + \nabla \cdot \left( \frac{k_{ww}(S_{ww})}{\mu_{ww}} K \nabla P \right) = 0.
\]
We assume that $k_{ww} = k_{wp}$ and that the functions are linear with respect to $c$. 
Effective water viscosity

- We assume that $k_{ww} = k_{wp}$ and that the functions are linear with respect to $c$, that is,

\[
\begin{align*}
    k_{wp}(S_{wp}) &= k_{wp}\left(\frac{c}{c_{\text{max}}}S_w\right) = \frac{c}{c_{\text{max}}}k_w(S_w), \\
    k_{ww}(S_{ww}) &= k_{ww}\left((1 - \frac{c}{c_{\text{max}}}S_w\right) = (1 - \frac{c}{c_{\text{max}}}k_w(S_w).
\end{align*}
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\end{align*}$$

We sum up the two equations of mass conservation ...

$$\begin{align*}
    \frac{\partial}{\partial t}(\phi S_{wp}) + \nabla \cdot \left(\frac{k_{wp}(S_{wp})}{\mu_{wp}} K \nabla P\right) &= 0 \\
    \frac{\partial}{\partial t}(\phi S_{ww}) + \nabla \cdot \left(\frac{k_{ww}(S_{ww})}{\mu_{ww}} K \nabla P\right) &= 0
\end{align*}$$
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\end{align*}
\]

- ... and obtain

\[
\frac{\partial}{\partial t} (\phi S_w) + \nabla \cdot \left( \left( \frac{c}{c_{\text{max}}} \frac{1}{\mu_{wp}} + (1 - \frac{c}{c_{\text{max}}}) \frac{1}{\mu_{ww}} \right) k_w(S_w) K \nabla P \right) = 0
\]
Effective water viscosity

- We assume that \( k_{ww} = k_{wp} \) and that the functions are linear with respect to \( c \), that is,

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\begin{align*}
    k_{wp}(S_{wp}) &= k_{wp}\left(\frac{c}{c_{\text{max}}} S_w\right) = \frac{c}{c_{\text{max}}} k_w(S_w), \\
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\end{align*}
\]

- ... and obtain

\[
\frac{\partial}{\partial t} (\phi S_w) + \nabla \cdot \left( \left( \frac{1}{\mu_{w,\text{eff}}} \right) k_w(S_w) K \nabla P \right) = 0,
\]

The equation of mass conservation for water.
The mixing parameter $\omega$

- If $\mu_{ww}(c) = \mu_{wp}(c)$: The two phases are identical. No interaction, polymer is simply transported.
The mixing parameter $\omega$

- If $\mu_{ww}(c) = \mu_{wp}(c)$: The two phases are identical. No interaction, polymer is simply transported.
- Intermediate cases are defined by introducing a parameter $\omega$

\[
\begin{align*}
\mu_{ww}(c) &= \mu_m(c)^\omega \mu_w^{1-\omega} \\
\mu_{wp}(c) &= \mu_m(c)^\omega \mu_p^{1-\omega}
\end{align*}
\]

where $\mu_m$ denotes the concentration of the fully mixed solution.
We look at the one dimensional problem: 2x2 system of hyperbolic conservation laws, with given total flux.
A polymer slug

- We look at the one dimensional problem: 2x2 system of hyperbolic conservation laws, with given total flux.
- Let \( \kappa = \left( \frac{\mu_p}{\mu_w} \right)^{1-\omega} \).
We look at the one dimensional problem: $2 \times 2$ system of hyperbolic conservation laws, with given total flux.

Let $\kappa = \left( \frac{\mu_p}{\mu_w} \right)^{1-\omega}$. 

Initial data
We look at the one dimensional problem: 2x2 system of hyperbolic conservation laws, with given total flux.

Let \( \kappa = \left( \frac{\mu_p}{\mu_w} \right)^{1-\omega} \).
A polymer slug

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- Let $\kappa = \left( \frac{\mu_p}{\mu_w} \right)^{1-\omega}$.

For all $\omega$, the water front propagates at same speed.
A polymer slug

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- Let \( \kappa = \left( \frac{\mu_p}{\mu_w} \right)^{1-\omega} \).

Case \( \omega = 1 \), fully mixed.
A polymer slug

- We look at the one dimensional problem: 2x2 system of hyperbolic conservation laws, with given total flux.

- Let $\kappa = \left( \frac{\mu_p}{\mu_w} \right)^{1-\omega}$.

Case $\omega < 1$, rarefaction wave at the tail.
Other features

- Adsorption.

Finally, mass conservation equations:

\[
\frac{\partial}{\partial t} \left( b_w \phi S_w \right) + \nabla \cdot \left( b_w \vec{v}_w \right) = 0,
\]

\[
\frac{\partial}{\partial t} \left( b_o \phi S_o \right) + \nabla \cdot \left( b_o \vec{v}_o \right) = 0,
\]

\[
\frac{\partial}{\partial t} \left( b_w \phi S_w c \right) + \frac{\partial}{\partial t} \left( (1 - \phi) \hat{c}_a \right) + \nabla \cdot \left( b_w c \vec{v}_wp \right) = 0,
\]

with

\[
\vec{v}_w = -k_{rw} \mu_{w,\text{eff}}(c) \mathbf{R}_k(c) \mathbf{K}(\nabla p_w - \rho_w g \nabla z),
\]

\[
\vec{v}_wp = -k_{rwp} \mu_{p,\text{eff}}(c) \mathbf{R}_k(c) \mathbf{K}(\nabla p_w - \rho_w g \nabla z) = m(c) \vec{v}_w.
\]
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- Adsorption.
- Reduced permeability.
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- Reduced permeability.
- Dead pore space. There is a problem with the model in eclipse. We use an alternative one.

Finally, mass conservation equations:

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\begin{align*}
\frac{\partial}{\partial t} (b_w \varphi S_w) + \nabla \cdot (b_w \vec{v}_w) &= 0, \\
\frac{\partial}{\partial t} (b_o \varphi S_o) + \nabla \cdot (b_o \vec{v}_o) &= 0, \\
\frac{\partial}{\partial t} (b_w \varphi S_w c) + \frac{\partial}{\partial t} ((1 - \varphi_{\text{ref}}) \hat{c}_a) + \nabla \cdot (b_w \vec{v}_w p) &= 0,
\end{align*}
\]

with

\[
\vec{v}_w = -k_{rw} \mu_w, \\
\vec{v}_wp = -k_{rwp} \mu_p, \\
\hat{c}_a R_k(c_a) K(\nabla p_w - \rho_w g \nabla z) = m(c) \vec{v}_w.
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- Finally, mass conservation equations:

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\[
\frac{\partial}{\partial t} (b_o \phi S_o) + \nabla \cdot (b_o \vec{v}_o) = 0,
\]
\[
\frac{\partial}{\partial t} (b_w \phi S_w c) + \frac{\partial}{\partial t} ((1 - \phi_{\text{ref}}) \hat{c}^a) + \nabla \cdot (b_w c \vec{v}_{wp}) = 0,
\]

with

\[
\vec{v}_w = - \frac{k_{rw}}{\mu_{w,\text{eff}}(c) R_k(c^a)} K (\nabla p_w - \rho_w g \nabla z),
\]
\[
\vec{v}_{wp} = - \frac{k_{rwp}}{\mu_{p,\text{eff}}(c) R_k(c^a)} K (\nabla p_w - \rho_w g \nabla z) = m(c) \vec{v}_w.
\]
Discretization and Splitting

- Space discretization: Two point flux and upwind approximation.
Discretization and Splitting

- Space discretization: Two point flux and upwind approximation.
- Time discretization: Implicit.
Water and oil residual for the cell $i$:

$$R_{\alpha,i}(S_{n+1}^i, c_{n+1}^i) = b_{n+1}^i \phi_{n+1}^i S_{n+1}^i - b_n^i \phi_n^i S_n^i$$

$$+ \frac{\Delta t}{V_i} \sum_{\{j|v_{i,j}^{n+1}<0\}} f_{\alpha}(S_{n+1}^j, c_{n+1}^j) b_{ij}^{n+1} v_{ij}^{n+1}$$

$$+ \frac{\Delta t}{V_i} f_{\alpha}(S_{n+1}^i, c_{n+1}^i) \sum_{\{j|v_{i,j}^{n+1}>0\}} b_{ij}^{n+1} v_{ij}^{n+1},$$

for $\alpha = \{w, o\}$
Polymer residual (only advection part)

Polymer residual for the cell $i$:

$$R_{c,i}(S^{n+1}, c^{n+1}) = b_{i}^{n+1} \phi_{i}^{n+1} S_{i}^{n+1} c_{i}^{n+1} + \hat{c}^{a}(c_{i}^{n+1})(1 - \phi_{ref,i})$$

$$- \left( b_{i}^{n} \phi_{i}^{n} S_{i}^{n} c_{i}^{n} + \hat{c}^{a}(c_{i}^{n})(1 - \phi_{ref,i}) \right)$$

$$+ \frac{\Delta t}{V_{i}} \sum_{\{j|v_{i,j}^{n+1} < 0\}} m(c_{j}^{n+1})c_{j}^{n+1} f_{w}(S_{j}^{n+1}, c_{j}^{n+1}) b_{ij}^{n+1} v_{ij}^{n+1}$$

$$+ m(c_{i}^{n+1})c_{i}^{n+1} f_{w}(S_{i}^{n+1}, c_{i}^{n+1}) \frac{\Delta t}{V_{i}} \sum_{\{j|v_{i,j}^{n+1} > 0\}} b_{ij}^{n+1} v_{ij}^{n+1}$$

$$= 0.$$
We can solve the fully coupled system of equations. Everything is let to the Newton solver. The physics is lost (even if it may come back via preconditioners).
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Alternatively, we split the equations. Each splitting part corresponds to a well-identified physical phenomenon. We regain control on the computation process.
Assume no polymer, incompressible fluids. Then, mass conservation for each phase is given by

\[ \frac{\partial (\phi S_\alpha)}{\partial t} + \nabla \cdot (\lambda_\alpha (S_\alpha) \mathbf{K} \nabla P) = 0 \]
The pressure equation

- Assume no polymer, incompressible fluids. Then, mass conservation for each phase is given by

\[ \frac{\partial (\phi S_\alpha)}{\partial t} + \nabla \cdot (\lambda_\alpha(S_\alpha) K \nabla P) = 0 \]

- After summation, it yields

\[ \phi \frac{\partial}{\partial t} \left( \sum_\alpha S_\alpha - 1 \right) = - \frac{\partial \phi}{\partial t} \left( \sum_\alpha S_\alpha - 1 \right) \]

\[ - \left( \frac{\partial \phi}{\partial t} - \nabla \cdot \left( \sum_\alpha (\lambda_\alpha(S_\alpha))K \nabla P \right) \right) \]
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- The pressure enforces volume preservation.
Single cell problem

- Discrete pressure equation

\[
\frac{1}{b_{w,i}} R_{w,i}(S^n, c^n) + \frac{1}{b_{o,i}} R_{o,i}(S^n, c^n) = 0.
\]
Single cell problem

- Discrete pressure equation

\[
\frac{1}{b_{w,i}} R_{w,i}(S^n, c^n) + \frac{1}{b_{o,i}} R_{o,i}(S^n, c^n) = 0.
\]

- Transport equation (advection part). The single cell problem for the cell \( i \) is given by

\[
R_{w,i}(S_i^{n+1}, c_i^{n+1}) = b_i^{n+1} \phi_i^{n+1} S_i^{n+1} - b_i^n \phi_i^n S_i^n
\]

\[
+ \frac{\Delta t}{V_i} \sum_{\{j|v_{i,j}^{n+1}<0\}} f_w(S_j^{n+1}, c_j^{n+1}) b_{ij}^{n+1} v_{ij}^{n+1}
\]

\[
+ \frac{\Delta t}{V_i} f_w(S_i^{n+1}, c_i^{n+1}) \sum_{\{j|v_{i,j}^{n+1}>0\}} b_{ij}^{n+1} v_{ij}^{n+1}
\]

\[= 0\]
The single cell problem is the fundamental building block in an iterative transport solver.
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Definition: We say that the transport solver is uniformly stable if there exists a unique solution to the single cell problem for any time step $\Delta t$. 

Uniform stability
We demonstrate the proof of unconditional stability in the simplest case. We want to solve

\[ R_w(S, c) = 0, \quad R_c(S, c) = 0. \]
Simple case for advection

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\[ R_w(S, c) = 0, \quad R_c(S, c) = 0. \]

- We show that given \( c \), there exists unique \( S(c) \) such that

\[ R_w(S(c), c) = 0 \]

- We check that

\[ R_w(0, c) \leq 0, \quad R_w(1, c) \geq 0 \] and \( \partial R_w / \partial S > 0 \).

- We look at function \( c \mapsto R_c(S(c), c) \) and check that:

\[
\begin{cases}
R_c(S(0), 0) \leq 0, \\
R_c(S(c_{\text{max}}), c_{\text{max}}) \geq 0, \\
\frac{d}{dc}(R_c(S(c), c)) > 0.
\end{cases}
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We check that \( R_w(0, c) \leq 0, R_w(1, c) \geq 0 \) and \( \frac{\partial R_w}{\partial S} > 0 \).

\[ R_w(0, c) = -\phi_i^n S_i^n + \frac{\Delta t}{V_i} \sum_{\{j|v_{i,j}^{n+1} < 0\}} f_w(S_{j}^{n+1}, c_{j}^{n+1})v_{i,j}^{n+1} \leq 0. \]

\[ R_w(1, c) = R_w,i(1, c) - (R_w,i(S^n, c^n) + R_o,i(S^n, c^n)) \]
\[ = \phi_i^n S_o,i^n - \frac{\Delta t}{V_i} \sum_{\{j|v_{i,j}^{n+1} < 0\}} (1 - f_w(S_{j}^{n+1}, c_{j}^{n+1}))v_{i,j}^{n+1} \geq 0 \]
Simple case for advection

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\begin{cases}
R_c(S(0), 0) \leq 0, \\
R_c(S(c_{\text{max}}), c_{\text{max}}) \geq 0, \\
\frac{d}{dc} (R_c(S(c), c)) > 0.
\end{cases}
\]

- We use essentially that

\[
\frac{\partial f_w}{\partial S} \geq 0 \quad \text{and} \quad \frac{\partial f_w}{\partial c} \leq 0
\]
We can prove that the single cell transport solver is uniformly stable for

- Compressible fluids, if $c_w \leq c_o$. In this case, we cannot expect $R_o = 0$ (mass conservation for oil).
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- Dead for space (if correctly introduced).
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- Dead for space (if correctly introduced).
- Adsorption.
- Reduced permeability.
The segregation equation is given by

\[
\frac{\partial}{\partial t} (\phi S_w) + \nabla \cdot \left( \frac{\lambda_w(S_w)\lambda_o(S_o)}{\lambda_w(S_w) + \lambda_o(S_o)} (\rho_o - \rho_w) gK \nabla z \right) = 0
\]
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\]

The residual for water mass conservation is given by
\[
R_w(S) = \phi_i (S_i - S_i^*) + \frac{g \Delta t}{V_i} \left( F(S_i, S_i+1) T_{i,i+1} (z_{i+1} - z_i) - F(S_{i-1}, S_i) T_{i-1,i} (z_i - z_{i-1}) \right) = 0
\]

where \( F(S_u, S_l) \) approximates the flux for segregation,
\[
F(S, S) = \frac{\lambda_w(S) \lambda_o(1 - S)}{\lambda_w(S) + \lambda_o(1 - S)}
\]
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\]

Upwind phase mobility:
\[
F(S_u, S_l) = \frac{\lambda_w(S_u) \lambda_o(1 - S_l)}{\lambda_w(S_u) + \lambda_o(1 - S_l)}.
\]
Let \( \mathbf{u} = (S, c) \). The residuals take the form

\[
\phi_i(S_i - S_i^*) + \frac{g \Delta t}{V_i} \left( F(\mathbf{u}_i, \mathbf{u}_{i+1}) T_{i,i+1}(z_{i+1} - z_i) - F(\mathbf{u}_{i-1}, \mathbf{u}_i) T_{i-1,i}(z_i - z_{i-1}) \right) = 0
\]

and

\[
\phi_i(S_ic_i - S_i^*c_i^*) + \frac{g \Delta t}{V_i} \left( G(\mathbf{u}_i, \mathbf{u}_{i+1}) T_{i,i+1}(z_{i+1} - z_i) - G(\mathbf{u}_{i-1}, \mathbf{u}_i) T_{i-1,i}(z_i - z_{i-1}) \right) = 0
\]
Let \( \mathbf{u} = (S, c) \). The residuals take the form

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\]

and

\[
\phi_i(S_i c_i - S_i^* c_i^*) + \frac{g \Delta t}{V_i} \left( G(\mathbf{u}_i, \mathbf{u}_{i+1}) T_{i,i+1}(z_{i+1} - z_i) - G(\mathbf{u}_{i-1}, \mathbf{u}_i) T_{i-1,i}(z_i - z_{i-1}) \right) = 0
\]

We can prove that the fluxes given by

\[
F(\mathbf{u}_u, \mathbf{u}_l) = \frac{\lambda_w(S_u, c_u) \lambda_o(1 - S_l)}{\lambda_w(S_u, c_u) + \lambda_o(1 - S_l)},
\]

\[
G(\mathbf{u}_u, \mathbf{u}_l) = m(c_u) c_u \frac{\lambda_w(S_u, c_l) \lambda_o(1 - S_l)}{\lambda_w(S_u, c_l) + \lambda_o(1 - S_l)}.
\]

yield stability.