Adaptive Higher Order Discontinuous Galerkin Methods for Two-phase Flow in Porous Media

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Motivation

- simulation of petroleum reservoirs,

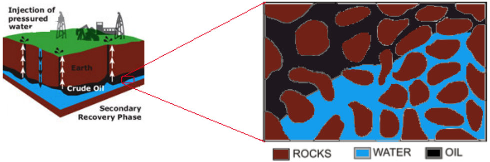


Abbildung: Oil recovery

- geological storage (CO2, nuclear wastes),

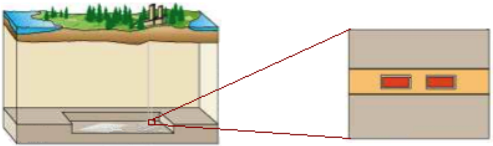


Abbildung: Nuclear waste repository

Motivation

DG methods for numerical simulations of complex geological systems

- Pros
  - High order convergence (depending on regularity)
  - Local conservation of physical quantities such as mass, momentum, and energy
  - Nonmatching grids, hp-adaptivity
  - Efficient use of memory hierarchy due to dense blocks

Motivation

DG methods for numerical simulations of complex geological systems

- Pros
  - High order convergence (depending on regularity)
  - Local conservation of physical quantities such as mass, momentum, and energy
  - Nonmatching grids, hp-adaptivity
  - Efficient use of memory hierarchy due to dense blocks
- Cons
  - Large number of degrees of freedom
  - ill-conditioning and denser global matrix with increasing approximation order

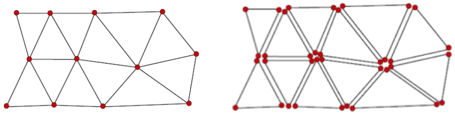


Abbildung: CG vs DG (dofs for piecewise linear)

Mathematical formulation for incompressible two phase flow

Domain $\Omega \in \mathbb{R}^d$, $d \in \{1,2,3\}$. The unknown variables are the phase pressures p_w, p_n and the phase saturations s_w, s_n.

- Phases={w,n}
  - both phases incompressible,
  - no dissolution.
- Model might include gravity,
- consider media heterogeneities.

The Darcy velocity for each phase is given by:

Darcy velocity

$$v_\alpha = -\lambda_\alpha K(\nabla p_\alpha - \rho_\alpha g), \; \alpha = \{w,n\} \tag{1}$$

where λ_α is the phase mobility, K is the permeability of the porous medium, ρ_α is the phase density, and g is the constant gravitational vector.

Phases mobilities

- Phases mobilities $\lambda_w = \lambda_w(s_n) = \frac{k_{rw}(s_n)}{\mu_w}$, $\qquad \lambda_n = \lambda_n(s_n) = \frac{k_{rn}(s_n)}{\mu_n}$

  where μ_α is the viscosity and $k_{r\alpha}$ is the relative permeability of phase $\alpha = \{w,n\}$.

Mathematical formulation

The balance of mass for each phase yields the saturation equation:

Balance of mass

$$\phi \frac{\partial \rho_\alpha s_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha v_\alpha) = \rho_\alpha q_\alpha, \quad \alpha \in \{w, n\} \tag{2}$$

where ϕ is the porosity, q_α is a source/sink term.

In addition to (2) and (1) closure relations must also be satisfied:

$$s_w + s_n = 1 \tag{3}$$

$$p_n - p_w = p_c \tag{4}$$

where $p_c = p_c(s_w)$ is the capillary pressure.

Mathematical formulation

Wetting phase pressure/non wetting phase saturation formulation

The unknowns are p_w and s_n.

$$-\nabla\cdot(\lambda_t K\nabla p_w + \lambda_n K\nabla p_c) = q_w + q_n \tag{5}$$

$$-\phi\frac{\partial s_n}{\partial t} - \nabla\cdot(\lambda_w K\nabla p_w) = q_w \tag{6}$$

Global pressure/nonwetting phase saturation formulation

The unknowns are p and s_n.

$$-\nabla\cdot(\lambda_t K\nabla p) = q_w + q_n \tag{7}$$

$$-\phi\frac{\partial s_n}{\partial t} - \nabla\cdot(\lambda_w K\nabla p - \lambda_n f_w K\nabla p_c) = q_w \tag{8}$$

the global pressure p is defined by:

$$p = p_n - \int_{1-s_{nr}}^{1-s_n} f_w p_c' + p_c(1-s_{nr}) \tag{9}$$

Mathematical formulation

p_c-p_w formulation

$$
\begin{aligned}
-\nabla\cdot[(\lambda_w+\lambda_n)K\nabla p_w+\lambda_n K\nabla p_c-(\rho_w\lambda_w+\rho_n\lambda_n)Kg]&=q_w+q_n,\\
\phi\frac{\partial\Psi(p_c)}{\partial t}-\nabla\cdot[\lambda_n K(\nabla p_w-\rho_n g)]-\nabla\cdot[\lambda_n K\nabla p_c]&=q_n.
\end{aligned}
\tag{10}
$$

Here $\lambda_\alpha=\lambda_\alpha(p_c),\ \alpha\in\{n,w\}$.

Saturation can be computed at given position x: $s_n(x,t)=\Psi(p_c(x,t))$.

Mathematical formulation

s_n-p_w formulation

Two coupled equations for p_w; s_n :

$$\begin{aligned} -\nabla\cdot[(\lambda_w+\lambda_n)K\nabla p_w+\lambda_n K\nabla p_c-(\rho_w\lambda_w+\rho_n\lambda_n)Kg] &= q_w+q_n, \\ \phi\frac{\partial s_n}{\partial t}-\nabla\cdot[\lambda_n K(\nabla p_w-\rho_n g)]-\nabla\cdot[\lambda_n K\nabla p_c] &= q_n. \end{aligned} \tag{11}$$

Here ϕ is the porosity, K is the permeability and q_w, q_n are source/sink term.

Non linearities

- Capillary pressure $p_c = p_c(s_n)$,
- Phases mobilities $\lambda_w = \lambda_w(s_n) = \frac{k_{rw}(s_n)}{\mu_w}$, $\quad \lambda_n = \lambda_n(s_n) = \frac{k_{rn}(s_n)}{\mu_n}$

  where μ_α is the viscosity and $k_{r\alpha}$ is the relative permeability of phase $\alpha = \{w,n\}$.

Non linearities

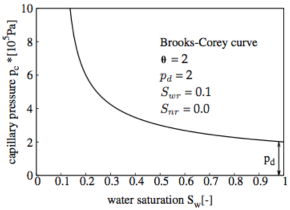
Non linearities

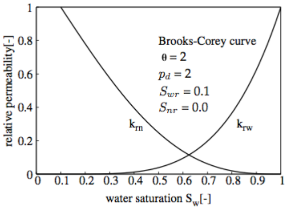
$$p_c(s_n) = p_d s_{e_w}^{\frac{-1}{\theta}}, \quad k_{rw}(s_{e_w}) = s_{e_w}^{\frac{2+3\theta}{\theta}}, \quad k_{rn}(s_{e_n}) = (s_{e_n})^2(1-(1-s_{e_n})^{\frac{2+\theta}{\theta}}), \tag{12}$$

where the effective saturation s_{e_α} is

$$s_{e_\alpha} = \frac{s_\alpha - s_{r\alpha}}{1 - s_{rw} - s_{rn}}, \quad \forall \alpha \in \{w,n\}. \tag{13}$$

Here $s_{r\alpha}$, $\alpha \in \{w,n\}$ are the phases residual saturations, $\theta \in [0.2, 3.0]$ is the inhomogeneity and $p_d \geq 0$ is the constant entry pressure.





Boundary conditions & initial values

Boundary divided into disjoint open sets $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$.

Boundary & initial conditions

$$s_n(x,0) = s_n^0(x),\ p_w(x,0) = p_w^0(x) \qquad \forall x \in \Omega \tag{14}$$

$$p_w(x,t) = p_{w_D}(x,t),\ s_n(x,t) = s_{n_D}(x,t) \qquad \forall x \in \Gamma_D \tag{15}$$

$$\rho_\alpha v_\alpha \cdot n = J_\alpha(x,t),\ J_t = \sum_{\alpha\in\{w,n\}} J_\alpha \qquad \forall x \in \Gamma_N \tag{16}$$

Here $J_\alpha,\ \alpha \in \{w,n\}$ is the inflow. Here n the outward normal to $\partial\Omega$ and $J_\alpha,\ \alpha \in \{w,n\}$ is the inflow.

Previous Work on DG for Two-Phase Flow

- Bastian & Riviere 2004
  - Global pressure / saturation formulation, splitting
  - Implicit/explicit saturation(+limiters), H(div)-projection

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  - no media discontinuities, no gravity, very coarse grids

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- Bastian 2013
  - Fully- coupled, higher-order in time, pw; pc formulation,
  - Media discontinuities, 1-3d

DG Finite element space

Domain Ω is subdivided into a partition $\mathcal{T}_h = \{E\}$ consisting of N_h elements.

DG Finite element space

The discontinuous finite element space is:
$\mathcal{D}_r(\mathcal{T}_h) = \{v \in \mathbb{L}^2(\Omega) : v_{|E} \in \mathbb{P}_r(E) \;\; \forall E \in \mathcal{T}_h\}$, with $\mathbb{P}_r(E)$ the space of polynomial functions of degree at most $1 \leq r$ on E.

- r_p for the pressure,
- r_s for the saturation.

Jump & Weighted average operator

Different types of domain can meet closely and cause large jumps in permeability.

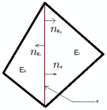

Abbildung: Two neigboring cells

Abbildung: Average & Jump

Jump & Weighted average operators

The jump is:

$$[\![p]\!] = p_{E_1} - p_{E_2}. \tag{17}$$

The weighted average:

$$\{p\}_\omega = \omega_{E_1} p_{E_1} + \omega_{E_2} p_{E_2}. \tag{18}$$

Here $\omega_{E_1} = \frac{\delta_K^{E_1}}{\delta_K^{E_1} + \delta_K^{E_2}}$ and $\omega_{E_2} = \frac{\delta_K^{E_2}}{\delta_K^{E_1} + \delta_K^{E_2}}$ with $\delta_K^{E_1} = n_e^T K_{E_2} n_e$ and $\delta_K^{E_2} = n_e^T K_{E_1} n_e$, K_{E_1} and K_{E_2} are the absolute permeabilities for E_1 and E_2.

Semi discrete formulation

The semi discrete weak formulation consist in finding the continuous in time approximations $p_{w,h}(\cdot,t) \in \mathscr{D}_{r_p}(\mathscr{T}_h)$, $s_{n,h}(\cdot,t) \in \mathscr{D}_{r_s}(\mathscr{T}_h)$ such that:

Semi discrete weak formulation

$$\begin{aligned} \mathscr{B}_h(p_{w,h};s_{n,h},v) = l_h(v) \quad & \forall v \in \mathscr{D}_{r_p}(\mathscr{T}_h),\ \forall t \in \mathscr{J}, \\ (\Phi\partial_t s_{n,h},z) + c_h(p_{w,h};s_{n,h},z) + d_h(s_{n,h},z) = r_h(z) \quad & \forall z \in \mathscr{D}_{r_s}(\mathscr{T}_h),\ \forall t \in \mathscr{J}. \end{aligned} \tag{19}$$

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The bilinear form $\mathcal{B}_h$ in the total fluid conservation equation (19) is expressed as:

$$\mathcal{B}_h(p_{w,h};s_{n,h},v) = \mathcal{B}_{bulk,h} + \mathcal{B}_{cons,h} + \mathcal{B}_{sym,h} + \mathcal{B}_{stab,h}. \tag{20}$$

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The first term $\mathscr{B}_{bulk,h}$ of (29) is the volume contribution:

$$\mathscr{B}_{bulk,h} := \mathscr{B}_{bulk,h}(p_{w,h},v;s_{n,h}) = \sum_{E \in \mathscr{T}_h} \int_E (\lambda_t K \nabla p_{w,h} + \lambda_n K \nabla p_{c,h} - (\rho_n \lambda_n + \rho_w \lambda_w) K g) \cdot \nabla v. \tag{21}$$

Total fluid conservation equation

$$\mathscr{B}_h(p_{w,h}; s_{n,h}, v) = \mathscr{B}_{bulk,h} + \mathscr{B}_{cons,h} + \mathscr{B}_{sym,h} + \mathscr{B}_{stab,h} = l_h(v).$$

The second term $\mathscr{B}_{cons,h}$, is the consistency term:

$$\begin{aligned} \mathscr{B}_{cons,h} := \mathscr{B}_{cons,h}(p_{w,h}, v; s_{n,h}) = &- \sum_{e \in \Gamma^h \cup \Gamma_D^h} \int_e \{\lambda_t K \nabla p_{w,h}\}_\omega \cdot n_e [\![v]\!] \\ &- \sum_{e \in \Gamma^h \cup \Gamma_D^h} \int_e \{\lambda_n K \nabla p_{c,h}\}_\omega \cdot n_e [\![v]\!] \\ &+ \sum_{e \in \Gamma^h \cup \Gamma_D^h} \int_e \{(\rho_n \lambda_n + \rho_w \lambda_w) K g)\}_\omega \cdot n_e [\![v]\!]. \end{aligned} \tag{22}$$

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The third term $\mathscr{B}_{sym,h}$, is the symmetry term. Depending on the choice of ϵ we get different DG methods ($\epsilon = -1$ SIPG, $\epsilon = 1$ NIPG, $\epsilon = 0$ IIPG):

$$\begin{aligned}\mathscr{B}_{sym,h} := \mathscr{B}_{sym,h}(p_{w,h},v;s_{n,h}) = &\epsilon \sum_{e\in\Gamma^h\cup\Gamma_D^h} \int_e \{\lambda_t K \nabla v \cdot n_e\}_\omega [\![p_{w,h}]\!] \\ &+ \epsilon \sum_{e\in\Gamma^h\cup\Gamma_D^h} \int_e \{\lambda_n K \nabla v \cdot n_e\}_\omega [\![s_{n,h}]\!].\end{aligned} \tag{23}$$

Total fluid conservation equation

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The last term $\mathscr{B}_{stab,h}$ is the stability term:

$$\mathscr{B}_{stab,h} := \mathscr{B}_{stab,h}(p_{w,h}, v) = \sum_{e \in \Gamma^h \cup \Gamma^h_D} \gamma^p_e \int_e [\![p_{w,h}]\!] [\![v]\!]. \tag{24}$$

We use in this work, unless specified otherwise, the following penalty formulation.

$$\gamma^p_e = C_p \frac{p(p+d-1)\,|\,e\,|}{min(|\,E_-\,|, |\,E_+\,|)}, \; c_p \geq 0 \tag{25}$$

Total fluid conservation equation

$$\mathscr{B}_h(p_{w,h};s_{n,h},v) = \mathscr{B}_{bulk,h} + \mathscr{B}_{cons,h} + \mathscr{B}_{sym,h} + \mathscr{B}_{stab,h} = l_h(v).$$

The right hand side of the total fluid conservation equation (19) is a linear form including the Neumann and Dirichlet boundary conditions and the source terms.

$$\begin{aligned} l_h(v) = &\int_\Omega (q_w + q_n)v - \sum_{e\in\Gamma_N} \int_e J_t v + \epsilon \sum_{e\in\Gamma_D^h} \int_e \lambda_t K \nabla v \cdot n_e p_D \\ &+ \epsilon \sum_{e\in\Gamma_D^h} \int_e \lambda_n K \nabla v \cdot n_e s_D + l_{stab}, \qquad \forall v \in \mathscr{D}_{r_p}(\mathscr{T}_h). \end{aligned} \tag{26}$$

Here $l_{stab}(v)$ is the stability term for the linear form:

$$l_{stab}(v) = \sum_{e\in\Gamma_D^h} \gamma_e^p \int_e p_D v \tag{27}$$

Phase conservation equation

$$(\Phi\partial_t s_{n,h}, z) + c_h(p_{w,h}; s_{n,h}, z) + d_h(s_{n,h}, z) = r_h(z) \qquad \forall z \in \mathscr{D}_{r_s}(\mathscr{T}_h),\ \forall t \in \mathscr{J}.$$

The convection term $-\nabla\cdot(\lambda_n K(\nabla p_w - \rho_n g))$ might be approximated by an upwind discretization technique.

$$\begin{aligned} c_h(p_{w,h}, z; s_{n,h}) = &\sum_{E\in\mathscr{T}_h}\int_E (K\lambda_n(\nabla p_{w,h} - \rho_n g))\cdot\nabla z - \sum_{e\in\Gamma^h\cup\Gamma^h_D}\int_e \{K\lambda_n^{\#}\nabla p_{w,h}\}_\omega \cdot n_e [\![z]\!] \\ &+ \sum_{e\in\Gamma^h\cup\Gamma^h_D}\int_e \{\rho_n K\lambda_n^{\#} g\}_\omega\cdot n_e [\![z]\!] + \epsilon \sum_{e\in\Gamma^h\cup\Gamma^h_D}\int_e \{K\lambda_n^{\#}\nabla z\}_\omega\cdot n_e [\![p_{w,h}]\!] \end{aligned} \tag{28}$$

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where $\lambda_n^{\#} = (1-\beta)\lambda_{n,E} + \beta\lambda_n^{\uparrow}$ and $\lambda_n^{\uparrow}$ is the upwind mobility:

$$\forall e \in \partial E_- \cap \partial E_+,\ \ \lambda_n^{\uparrow} = \begin{cases} \lambda_{n,E_+} & \text{if } -K(\nabla p_w + \nabla p_c - \rho_n g)\cdot n \geq 0, \\ \lambda_{n,E_-} & \text{else.} \end{cases}$$

Hence depending on the value of $\beta \in \{0,1\}$, we might use central differencing or upwinding of the mobility for internal interfaces.

Phase conservation equation

$$(\Phi\partial_t s_{n,h}, z) + c_h(p_{w,h}; s_{n,h}, z) + d_h(s_{n,h}, z) = r_h(z) \qquad \forall z \in \mathscr{D}_{r_s}(\mathscr{T}_h),\ \forall t \in \mathscr{J}.$$

The diffusion term $-\nabla\cdot(\lambda_n K \nabla p_c)$ is discretized by a bilinear form similar to that of (29).

$$\begin{aligned} d_h(s_{n,h}, z) = & \sum_{E\in\mathscr{T}_h} \int_E \lambda_n K \nabla p_{c,h} \cdot \nabla z - \sum_{e\in\Gamma^h\cup\Gamma_D^h} \int_e \{\lambda_n K \nabla p_{c,h}\}_\omega \cdot n_e [\![z]\!] \\ & + \epsilon \sum_{e\in\Gamma^h\cup\Gamma_D^h} \int_e \{\lambda_n K \nabla z\}_\omega \cdot n_e [\![s_{n,h}]\!] + \sum_{e\in\Gamma^h\cup\Gamma_D^h} \gamma_e^s \int_e [\![s_{n,h}]\!] [\![z]\!]. \end{aligned} \tag{29}$$

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$$\begin{aligned} d_h(s_{n,h}, z) = &\sum_{E\in\mathscr{T}_h}\int_E \lambda_n K\nabla p_{c,h}\cdot\nabla z - \sum_{e\in\Gamma^h\cup\Gamma_D^h}\int_e \{\lambda_n K\nabla p_{c,h}\}_\omega\cdot n_e [\![z]\!] \\ &+\epsilon\sum_{e\in\Gamma^h\cup\Gamma_D^h}\int_e \{\lambda_n K\nabla z\}_\omega\cdot n_e [\![s_{n,h}]\!] + \sum_{e\in\Gamma^h\cup\Gamma_D^h}\gamma_e^s\int_e [\![s_{n,h}]\!][\![z]\!]. \end{aligned} \tag{29}$$

The right hand side r_h includes the Neumman and Dirichlet boundary condition and the nonwetting source term.

$$\begin{aligned} r_h(z) = &\int_\Omega q_n z - \sum_{e\in\Gamma_N}\int_e J_n z + \epsilon\sum_{e\in\Gamma_D^h}\int_e \lambda_n K\nabla z\cdot n_e p_D \\ &+\epsilon\sum_{e\in\Gamma_D^h}\int_e \lambda_n K\nabla z\cdot n_e p_c(s_D) + \sum_{e\in\Gamma_D^h}\gamma_e^s\int_e s_D z, \qquad \forall z\in\mathscr{D}_{r_s}(\mathscr{T}_h). \end{aligned} \tag{30}$$

where:

$$\gamma_e^s = C_s\frac{p(p+d-1)|e|}{min(|E_-|,|E_+|)},\ c_s \geq 0. \tag{31}$$

Fully discrete formulation

Application of Backward Euler time discretization and Interior Penalty DG for space discretization to the system of PDE,s for the coupled equations (19) - (19) yields:

Fully discrete formulation

$$\mathscr{B}_h(p_{w,h}^{i+1}; s_{n,h}^{i+1}, v) = l_h(v), \qquad \forall\, v \in \mathscr{D}_{r_p}(\mathscr{T}_h), \tag{32}$$

$$(\Phi \frac{s_{n,h}^{i+1} - s_{n,h}^{i}}{\Delta t}, z) + c_h(p_{w,h}^{i+1}; s_{n,h}^{i+1}, z) + d_h(s_{n,h}^{i+1}, z) = r_h(z), \qquad \forall\, z \in \mathscr{D}_{r_s}(\mathscr{T}_h), \tag{33}$$

$$(s_{n,h}^{0}, \phi) = (s_n^0, \phi), \qquad \forall\, \phi \in \mathscr{D}_{r_s}(\mathscr{T}_h). \tag{34}$$

A posteriori estimator & adaptivity state of work

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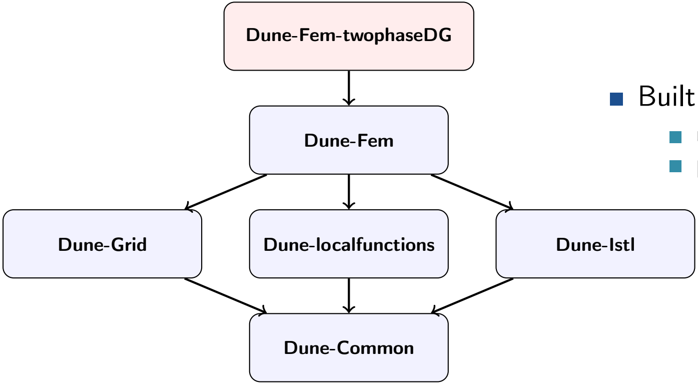
  - coupled diffusion and advection dominated transport
- Vohralik & Wheeler (2013)
  - general abstract framework,
  - a posteriori estimates for immiscible incom- pressible two-phase flows in porous media.

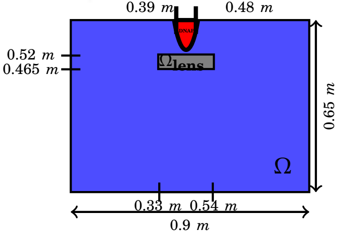
Software framework

Challenge: Develop efficients implementation with DG on a sustainable framework.



- Built on top of Dune-Fem
  - user-friendly implementation,
  - profit from many features,
    - DG discretization spaces,
    - efficient solvers and grids,
    - support for parallelization and adaptivity.

$2d$ infiltration problem



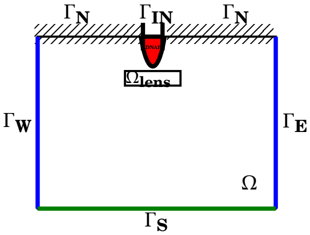


Abbildung: Geometry and boundary conditions for the DNAPL infiltration problem

	Ω_{lens}	$\Omega\backslash\Omega_{lens}$
Φ [-]	0.39	0.40
k [m^2]	6.64×10^{-16}	6.64×10^{-11}
S_{wr} [-]	0.1	0.12
S_{nr} [-]	0.00	0.00
θ [-]	2.0	2.70
p_d [Pa]	5000	755

Tabelle: Parameters

Γ_{IN}	$J_n=-0.075,\ J_w=0$
Γ_N	$J_n=0.00,\ J_w=0.00$
Γ_S	$J_w=0,\ J_n=0.00$
$\Gamma_E\cup\Gamma_W$	$p_w=(0.65-y)\cdot9810,\ s_n=0$

Tabelle: Boundary conditions

$2d$ infiltration problem

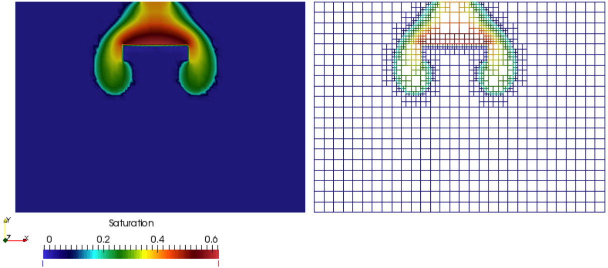


Abbildung: DNAPL saturation distribution after 2000 s (left), leaf grid (right). Polynomial order $p = 3$.

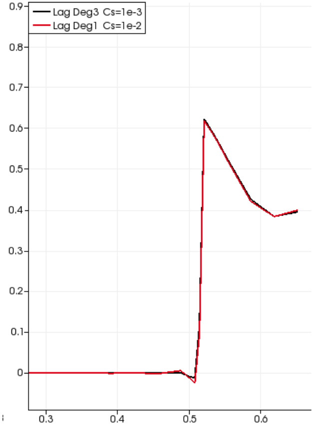


Abbildung: Comparison of non-wetting-phase saturation at T=2000 s. Left, profile along the line ((0,0);(0.45,0.65)). Right, profile along the line x=0.45 m.

$3d$ infiltration problem

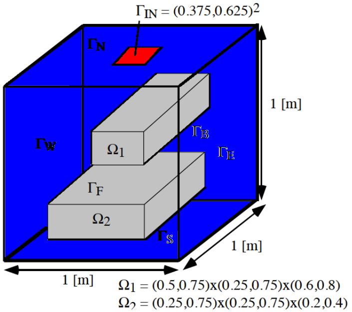


Abbildung: Geometry of the domain for the 3d DNAPL infiltration problem

	Ω_1	Ω_2	$\Omega \setminus \Omega_1 \cap \Omega \setminus \Omega_2$
Φ [-]	0.39	0.39	0.40
k $[m^2]$	6.64×10^{-16}	6.64×10^{-15}	6.64×10^{-11}
S_{wr} [-]	0.1	0.1	0.12
S_{nr} [-]	0.00	0.00	0.00
θ [-]	2.0	2.0	2.70
p_d [Pa]	5000	5000	755

Abbildung: 3d problem parameters

$3d$ infiltration problem

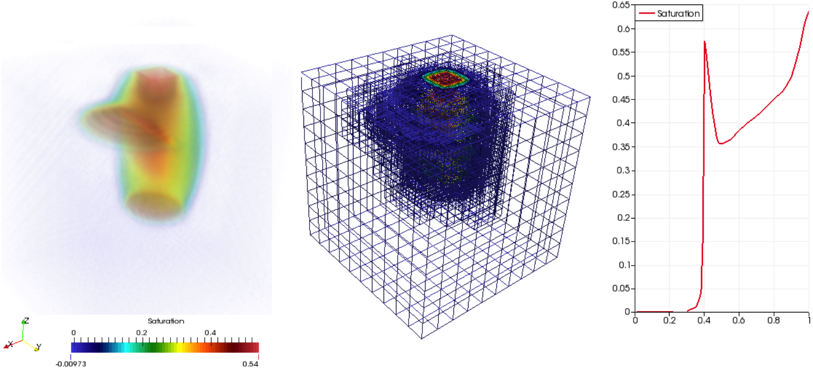


Abbildung: Contour plot of saturation distribution after 3600 s of injection with 0.25 $Kg\ s^{-1}m^{-2}$ of DNAPL in a depth of 1 m (left column), leaf grid (center column) and saturation profile along the line ((1,0.45,0);(0.45,0.45,1))(right column).

$3d$ infiltration problem

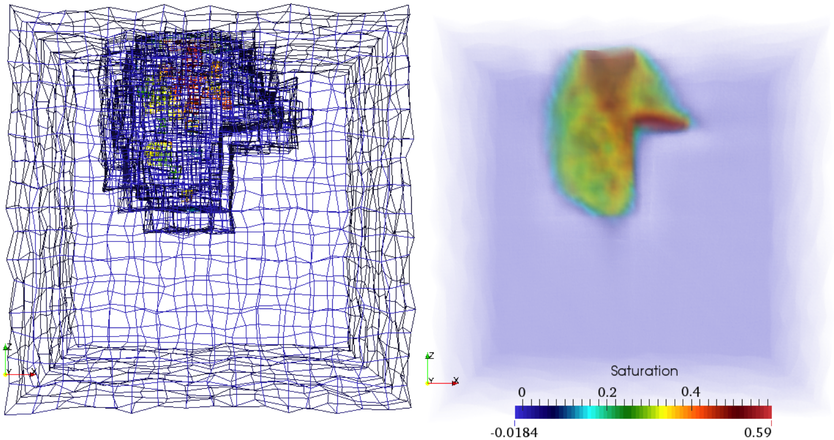


Abbildung: 3d-Problem: Saturation distribution after 3000 s of injection with 0.25 $Kg\ s^{-1}m^{-2}$ of DNAPL. Unstructured mesh. Polynomial order $p=1$.

Test case with analytical solution

We consider a system of partial differential equations with known exact solution.

Problem

Considering $\Omega = (0,1)^2$ and $J = (0,T)$, find (p,s) such that

$$-\nabla \cdot (\lambda(s) K \nabla p) = 0 \qquad \text{in } \Omega \times J \tag{35}$$

$$\phi \frac{\partial s}{\partial t} - \nabla \cdot (-\epsilon \nabla s + f(s)\lambda(s) K \nabla p) = q \qquad \text{in } \Omega \times J \tag{36}$$

with $\lambda(s) = (0.5 - 0.2s)^{-1}$, $\epsilon = 0.01$, $f(s) = s$, where $q = 2\pi\epsilon \sin(\pi(x_1 + x_2 - 2t))$

Boundary & Initial conditions

$$p(x,y,t) = \frac{0.2}{\pi}\cos(\pi(x+y-2t)) - 0.5(x+y), \quad s(x,y,t) = \sin(\pi(x+y-2t)) \quad \forall (x,y,t) \in \partial\Omega \times J$$

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Test case with analytical solution

$AvgNb_{dofs}$	$MaxNb_{dofs}$	$\|p - p_h\|_{L^2(\Omega)}$	$\|s - s_h\|_{L^2(\Omega)}$
665.15	792	1.110^{-2}	$4.7\ 10^{-2}$
1944.47	2570	4.9410^{-3}	$2.87\ 10^{-2}$
3483	4624	3.7110^{-3}	$2.62\ 10^{-2}$

Tabelle: TYPEPARAM algorithm (200 time steps,T=0.2 [s], Newton tol 1e-6, residual tol 1e-7)

$AvgNb_{dofs}$	$MaxNb_{dofs}$	$\|p - p_h\|_{L^2(\Omega)}$	$\|s - s_h\|_{L^2(\Omega)}$
722	960	1.410^{-2}	$4.64\ 10^{-2}$
2583.1	3108	4.85410^{-3}	$3.29\ 10^{-2}$

Tabelle: ERNDOVO algorithm (200 time steps,T=0.2 [s], Newton tol 1e-6, residual tol 1e-7)

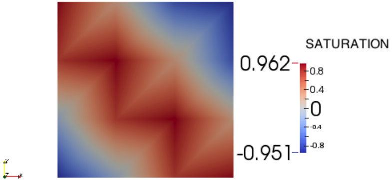

Abbildung: Saturation profiles at $T = 0.2$

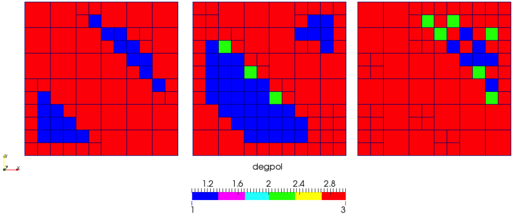

Abbildung: Polynomial degrees at $T = 0.2$

h-p test Case

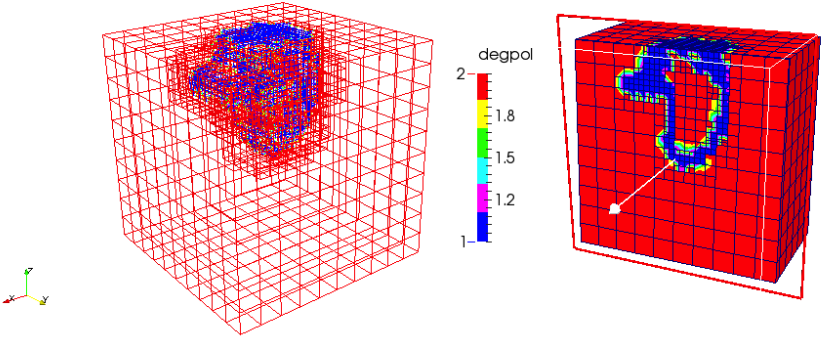

Abbildung: 3d-Problem: hp adaptive 3d problem mesh.Max polynomial order $p = 2$.

In summary

- Fully-coupled discontinuous Galerkin (DG) method for incompressible two-phase flow with discontinuous capillary pressure,
  - no post-processing of the DG velocity field in contrast to results for decoupled schemes
- Interior Penalty DG formulation,
  - Weighted averages,
- h, p and h-p adaptivity,
- Higher order polynomials up to piecewise cubics are implemented.

Thank you for your interest!