Adaptive Higher Order Discontinuous Galerkin Methods for Two-phase Flow in Porous Media

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- simulation of petroleum reservoirs,
- geological storage (CO2, nuclear wastes),
Motivation

DG methods for numerical simulations of complex geological systems

- **Pros**
  - High order convergence (depending on regularity)
  - Local conservation of physical quantities such as mass, momentum, and energy
  - Nonmatching grids, hp-adaptivity
  - Efficient use of memory hierarchy due to dense blocks
Motivation

DG methods for numerical simulations of complex geological systems

- **Pros**
  - High order convergence (depending on regularity)
  - Local conservation of physical quantities such as mass, momentum, and energy
  - Nonmatching grids, hp-adaptivity
  - Efficient use of memory hierarchy due to dense blocks

- **Cons**
  - Large number of degrees of freedom
  - Ill-conditioning and denser global matrix with increasing approximation order

Abbildung: CG vs DG (dofs for piecewise linear)
Mathematical formulation for incompressible two phase flow

Domain $\Omega \in \mathbb{R}^d$, $d \in \{1, 2, 3\}$. The unknown variables are the phase pressures $p_w, p_n$ and the phase saturations $s_w, s_n$.

- Phases=$\{w,n\}$
  - both phases incompressible,
  - no dissolution.

- Model might include gravity,
  - consider media heterogeneities.

The Darcy velocity for each phase is given by:

\[
v_\alpha = -\lambda_\alpha K(\nabla p_\alpha - \rho_\alpha g), \quad \alpha = \{w, n\}
\]

where $\lambda_\alpha$ is the phase mobility, $K$ is the permeability of the porous medium, $\rho_\alpha$ is the phase density, and $g$ is the constant gravitational vector.

Phases mobilities

- Phases mobilities $\lambda_w = \lambda_w(s_n) = \frac{k_{rw}(s_n)}{\mu_w}$, $\lambda_n = \lambda_n(s_n) = \frac{k_{rn}(s_n)}{\mu_n}$

where $\mu_\alpha$ is the viscosity and $k_{r\alpha}$ is the relative permeability of phase $\alpha = \{w, n\}$. 
Mathematical formulation

The balance of mass for each phase yields the saturation equation:

\[
\phi \frac{\partial \rho \alpha s \alpha}{\partial t} + \nabla \cdot (\rho \alpha v \alpha) = \rho \alpha q \alpha, \quad \alpha \in \{w, n\} \tag{2}
\]

where \( \phi \) is the porosity, \( q \alpha \) is a source/sink term.

In addition to (2) and (1) closure relations must also be satisfied:

\[
s_w + s_n = 1 \tag{3}
\]
\[
p_n - p_w = p_c \tag{4}
\]

where \( p_c = p_c(s_w) \) is the capillary pressure.
Mathematical formulation

Wetting phase pressure/non wetting phase saturation formulation

The unknowns are \( p_w \) and \( s_n \).

\[
- \nabla \cdot (\lambda_t K \nabla p_w + \lambda_n K \nabla p_c) = q_w + q_n
\]

(5)

\[
-\frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_w K \nabla p_w) = q_w
\]

(6)

Global pressure/nonwetting phase saturation formulation

The unknowns are \( p \) and \( s_n \).

\[
- \nabla \cdot (\lambda_t K \nabla p) = q_w + q_n
\]

(7)

\[
-\frac{\partial s_n}{\partial t} - \nabla \cdot (\lambda_w K \nabla p - \lambda_n f_w K \nabla p_c) = q_w
\]

(8)

the global pressure \( p \) is defined by:

\[
p = p_n - \int_{1-s_n}^{1-s_{nr}} f_w p'_c + p_c (1-s_{nr})
\]

(9)
Mathematical formulation

\[ p_c - p_w \] formulation

\[-\nabla \cdot [(\lambda_w + \lambda_n)K\nabla p_w + \lambda_n K\nabla p_c - (\rho_w \lambda_w + \rho_n \lambda_n)Kg] = q_w + q_n,
\]

\[ \phi \frac{\partial \Psi(p_c)}{\partial t} - \nabla \cdot [\lambda_n K(\nabla p_w - \rho_n g)] - \nabla \cdot [\lambda_n K\nabla p_c] = q_n. \]

Here \( \lambda_\alpha = \lambda_\alpha(p_c), \ \alpha \in \{n, w\} \).

Saturation can be computed at given position \( x \): \( s_n(x,t) = \Psi(p_c(x,t)) \).
Mathematical formulation

**$s_n$-$p_w$ formulation**

Two coupled equations for $p_w, s_n$:

\[
- \nabla \cdot \left[ (\lambda_w + \lambda_n)K \nabla p_w + \lambda_n K \nabla p_c - (\rho_w \lambda_w + \rho_n \lambda_n)Kg \right] = q_w + q_n,
\]

\[
\phi \frac{\partial s_n}{\partial t} - \nabla \cdot \left[ \lambda_n K (\nabla p_w - \rho_n g) \right] - \nabla \cdot \left[ \lambda_n K \nabla p_c \right] = q_n.
\]

(11)

Here $\phi$ is the porosity, $K$ is the permeability and $q_w, q_n$ are source/sink term.

**Non lineairities**

- **Capillary pressure** $p_c = p_c(s_n)$,
- **Phases mobilities** $\lambda_w = \lambda_w(s_n) = \frac{k_{rw}(s_n)}{\mu_w}$, $\lambda_n = \lambda_n(s_n) = \frac{k_{rn}(s_n)}{\mu_n}$

where $\mu_\alpha$ is the viscosity and $k_{r\alpha}$ is the relative permeability of phase $\alpha = \{w, n\}$. 
Non linearities

\[
p_c(s_n) = p_d s_{e_w}^{\frac{1}{\theta}}, \quad k_{rw}(s_{e_w}) = s_{e_w}^{\frac{2+3\theta}{\theta}}, \quad k_{rn}(s_{e_n}) = (s_{e_n})^2(1 - (1 - s_{e_n})^{\frac{2+\theta}{\theta}}),
\]

where the effective saturation \( s_{e\alpha} \) is

\[
s_{e\alpha} = \frac{s_{\alpha} - s_{r\alpha}}{1 - s_{rw} - s_{rn}}, \quad \forall \alpha \in \{w, n\}.
\]

Here \( s_{r\alpha}, \alpha \in \{w, n\} \) are the phases residual saturations, \( \theta \in [0.2, 3.0] \) is the inhomogeneity and \( p_d \geq 0 \) is the constant entry pressure.
Boundary conditions & initial values

Boundary divided into disjoint open sets \( \partial \Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N \).

### Boundary & initial conditions

\[
\begin{align*}
  s_n(x, 0) &= s_n^0(x), \quad p_w(x, 0) = p_w^0(x) & \forall x \in \Omega \\
p_w(x, t) &= p_{wD}(x, t), \quad s_n(x, t) = s_{nD}(x, t) & \forall x \in \Gamma_D \\
\rho_\alpha v_\alpha \cdot n &= J_\alpha(x, t), \quad J_t = \sum_{\alpha \in \{w, n\}} J_\alpha & \forall x \in \Gamma_N
\end{align*}
\]

(14) \hspace{1cm} (15) \hspace{1cm} (16)

Here \( J_\alpha, \alpha \in \{w, n\} \) is the inflow. Here \( n \) the outward normal to \( \partial \Omega \) and \( J_\alpha, \alpha \in \{w, n\} \) is the inflow.
Previous Work on DG for Two-Phase Flow

- Bastian & Riviere 2004
  - Global pressure / saturation formulation, splitting
  - Implicit/explicit saturation(+limiters), H(div)-projection
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- Kloefkorn 2009
  - Compact Discotinuous Galerkin method
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  - Splitting, global pressure, implicit saturation, H(div)-projection
  - Media discontinuities, 1D, no gravity

- **Bastian 2013**
  - Fully- coupled, higher-order in time, pw; pc formulation,
  - Media discontinuities, 1-3d
Domain $\Omega$ is subdivided into a partition $\mathcal{T}_h = \{E\}$ consisting of $N_h$ elements.

The discontinuous finite element space is:

$$\mathcal{D}_r(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_E \in \mathcal{P}_r(E) \quad \forall E \in \mathcal{T}_h\},$$

with $\mathcal{P}_r(E)$ the space of polynomial functions of degree at most $1 \leq r$ on $E$.

- $r_p$ for the pressure,
- $r_s$ for the saturation.
Jump & Weighted average operator

Different types of domain can meet closely and cause large jumps in permeability.

**Jump & Weighted average operators**

The jump is:
\[
\begin{bmatrix} p \end{bmatrix} = p_{E_1} - p_{E_2}.
\] (17)

The weighted average:
\[
\{ p \}_\omega = \omega_{E_1} p_{E_1} + \omega_{E_2} p_{E_2}.
\] (18)

Here \( \omega_{E_1} = \frac{\delta_{E_1} E_1}{\delta_{K} E_1 + \delta_{K} E_2} \) and \( \omega_{E_2} = \frac{\delta_{E_2} E_2}{\delta_{K} E_1 + \delta_{K} E_2} \) with \( \delta_{E_1} = n_e^T K_{E_1} n_e \) and \( \delta_{E_2} = n_e^T K_{E_1} n_e \). \( K_{E_1} \) and \( K_{E_2} \) are the absolute permeabilities for \( E_1 \) and \( E_2 \).
Semi discrete formulation

The semi discrete weak formulation consist in finding the continuous in time approximations $p_{w,h}(\cdot, t) \in \mathcal{D}_p(T_h)$, $s_{n,h}(\cdot, t) \in \mathcal{D}_s(T_h)$ such that:

\[
\mathcal{B}_h(p_{w,h}; s_{n,h}, v) = l_h(v) \quad \forall v \in \mathcal{D}_p(T_h), \forall t \in J, \quad (19)
\]

\[
\Phi \partial_t s_{n,h} + c_h(p_{w,h}; s_{n,h}, z) + d_h(s_{n,h}, z) = r_h(z) \quad \forall z \in \mathcal{D}_s(T_h), \forall t \in J.
\]
Semi discrete formulation

The semi discrete weak formulation consist in finding the continuous in time approximations \( p_{w,h}(\cdot,t) \in \mathcal{D}_r(T_h), s_{n,h}(\cdot,t) \in \mathcal{D}_s(T_h) \) such that:

\[
\mathcal{B}_h(p_{w,h};s_{n,h},v) = l_h(v) \quad \forall v \in \mathcal{D}_r(T_h), \quad \forall t \in J,
\]

\[
(\Phi \partial_t s_{n,h},z) + c_h(p_{w,h};s_{n,h},z) + d_h(s_{n,h},z) = r_h(z) \quad \forall z \in \mathcal{D}_s(T_h), \quad \forall t \in J.
\] (19)

The bilinear form \( \mathcal{B}_h \) in the total fluid conservation equation (19) is expressed as:

\[
\mathcal{B}_h(p_{w,h};s_{n,h},v) = \mathcal{B}_{\text{bulk},h} + \mathcal{B}_{\text{cons},h} + \mathcal{B}_{\text{sym},h} + \mathcal{B}_{\text{stab},h}.
\] (20)
Semi discrete formulation

The semi discrete weak formulation consist in finding the continuous in time approximations $p_{w,h}(\cdot,t) \in \mathcal{D}_r(T_h)$, $s_{n,h}(\cdot,t) \in \mathcal{D}_r(T_h)$ such that:

\[ \mathcal{B}_h(p_{w,h};s_{n,h},v) = l_h(v) \quad \forall v \in \mathcal{D}_r(T_h), \forall t \in \mathcal{I}, \]

\[ (\Phi \partial_t s_{n,h}, z) + c_h(p_{w,h};s_{n,h},z) + d_h(s_{n,h},z) = r_h(z) \quad \forall z \in \mathcal{D}_r(T_h), \forall t \in \mathcal{I}. \]

The bilinear form $\mathcal{B}_h$ in the total fluid conservation equation (19) is expressed as:

\[ \mathcal{B}_h(p_{w,h};s_{n,h},v) = \mathcal{B}_{bulk,h} + \mathcal{B}_{cons,h} + \mathcal{B}_{sym,h} + \mathcal{B}_{stab,h}. \]

The first term $\mathcal{B}_{bulk,h}$ of (29) is the volume contribution:

\[ \mathcal{B}_{bulk,h} := \mathcal{B}_{bulk,h}(p_{w,h},v;s_{n,h}) = \sum_{E \in T_h} \int_E (\lambda_t K \nabla p_{w,h} + \lambda_n K \nabla p_{c,h} - (\rho_n \lambda_n + \rho_w \lambda_w) Kg) \cdot \nabla v. \]
Total fluid conservation equation

\[ \mathcal{B}_h(p_{w,h};s_{n,h},v) = \mathcal{B}_{bulk,h} + \mathcal{B}_{cons,h} + \mathcal{B}_{sym,h} + \mathcal{B}_{stab,h} = l_h(v). \]

The second term \( \mathcal{B}_{cons,h} \), is the consistency term:

\[
\mathcal{B}_{cons,h} := \mathcal{B}_{cons,h}(p_{w,h},v;s_{n,h}) = - \sum_{e \in \Gamma^h \cup \Gamma_D^h} \int_e \{\lambda_t K \nabla p_{w,h}\}_{\omega} \cdot n_e[v]
- \sum_{e \in \Gamma^h \cup \Gamma_D^h} \int_e \{\lambda_n K \nabla p_{c,h}\}_{\omega} \cdot n_e[v]
+ \sum_{e \in \Gamma^h \cup \Gamma_D^h} \int_e \{(\rho_n \lambda_n + \rho_w \lambda_w) Kg\}_{\omega} \cdot n_e[v].
\]
Total fluid conservation equation

$$B_h(p_w,h; s_n,h,v) = B_{bulk,h} + B_{cons,h} + B_{sym,h} + B_{stab,h} = l_h(v).$$

The second term $B_{cons,h}$, is the consistency term:

$$B_{cons,h} := B_{cons,h}(p_w,h,v; s_n,h) = - \sum_{e \in \Gamma^h_D} \int_e \{\lambda_t K \nabla p_w,h\}_\omega \cdot n_e[v]$$

$$- \sum_{e \in \Gamma^h_D} \int_e \{\lambda_n K \nabla p_c,h\}_\omega \cdot n_e[v]$$

$$+ \sum_{e \in \Gamma^h_D} \int_e \{(\rho_n \lambda_n + \rho_w \lambda_w) Kg\}_\omega \cdot n_e[v].$$

The third term $B_{sym,h}$, is the symmetry term. Depending on the choice of $\epsilon$ we get different DG methods ($\epsilon = -1$ SIPG, $\epsilon = 1$ NIPG, $\epsilon = 0$ IIPG):

$$B_{sym,h} := B_{sym,h}(p_w,h,v; s_n,h) = \epsilon \sum_{e \in \Gamma^h_D} \int_e \{\lambda_t K \nabla v \cdot n_e\}_\omega [p_w,h]$$

$$+ \epsilon \sum_{e \in \Gamma^h_D} \int_e \{\lambda_n K \nabla v \cdot n_e\}_\omega [s_n,h].$$
Total fluid conservation equation

\[ B_h(p_w,h; s_n,h,v) = B_{bulk,h} + B_{cons,h} + B_{sym,h} + B_{stab,h} = l_h(v). \]

The last term \( B_{stab,h} \) is the stability term:

\[ B_{stab,h} := B_{stab,h}(p_w,h,v) = \sum_{e \in \Gamma^h \cup \Gamma^D} \gamma^p_e \int_e [p_w,h][v]. \] (24)

We use in this work, unless specified otherwise, the following penalty formulation.

\[ \gamma^p_e = C_p \frac{p(p + d - 1) |e|}{\min(|E_-|, |E_+|)}, \quad c_p \geq 0 \] (25)
The right hand side of the total fluid conservation equation (19) is a linear form including the Neumann and Dirichlet boundary conditions and the source terms.

\[
l_h(v) = \int_\Omega (q_w + q_n)v - \sum_{e \in \Gamma_N} \int_e J_t v + \epsilon \sum_{e \in \Gamma_D^h} \int_e \lambda_t K \nabla v \cdot n_e p_D \\
+ \epsilon \sum_{e \in \Gamma_D^h} \int_e \lambda_n K \nabla v \cdot n_e s_D + l_{\text{stab}}, \quad \forall v \in \mathcal{D}_r(T_h).
\]

(26)

Here \(l_{\text{stab}}(v)\) is the stability term for the linear form:

\[
l_{\text{stab}}(v) = \sum_{e \in \Gamma_D^h} \gamma_e^p \int_e p_D v
\]

(27)
Phase conservation equation

\[(\Phi \partial_t s_{n,h},z) + c_h(p_{w,h};s_{n,h},z) + d_h(s_{n,h},z) = r_h(z) \quad \forall z \in D_r(T_h), \forall t \in J.\]

The convection term \(-\nabla \cdot (\lambda_n K(\nabla p_w - \rho_n g))\) might be approximated by an upwind discretization technique.

\[
c_h(p_{w,h},z;s_{n,h}) = \sum_{E \in T_h} \int_E (K \lambda_n (\nabla p_w - \rho_n g)) \cdot \nabla z - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{K \lambda_n^\# \nabla p_w \}_\omega \cdot n_e [z] + \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\rho_n K \lambda_n^\# g \}_\omega \cdot n_e [z] + \epsilon \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{K \lambda_n^\# \nabla z \}_\omega \cdot n_e [p_{w,h}] \quad (28)
\]
Phase conservation equation

\[(\Phi \partial_t s_{n,h}, z) + c_h(p_{w,h}; s_{n,h}, z) + d_h(s_{n,h}, z) = r_h(z) \quad \forall z \in D_{rs}(\mathcal{F}_h), \forall t \in \mathcal{T}.
\]

The convection term \(-\nabla \cdot (\lambda_n K(\nabla p_w - \rho_n g))\) might be approximated by an upwind discretization technique.

\[
c_h(p_{w,h}, z; s_{n,h}) = \sum_{E \in \mathcal{F}_h} \int_E (K \lambda_n (\nabla p_w - \rho_n g)) \cdot \nabla z - \sum_{e \in \Gamma^h} \int_e \{K \lambda_n^\# \nabla p_w,h\} \omega \cdot n_e[z] + \sum_{e \in \Gamma^h} \int_e \{\rho_n K \lambda_n^\# g\} \omega \cdot n_e[z] + \epsilon \sum_{e \in \Gamma^h} \int_e \{K \lambda_n^\# \nabla z\} \omega \cdot n_e[p_{w,h}]
\]

(28)

where \(\lambda_n^\# = (1 - \beta)\lambda_{n,E} + \beta \lambda_n^\uparrow\) and \(\lambda_n^\uparrow\) is the upwind mobility:

\[
\forall e \in \partial E_0 \cap \partial E_+, \quad \lambda_n^\uparrow = \begin{cases} 
\lambda_{n,E} & \text{if } -K(\nabla p_w + \nabla p_c - \rho_n g) \cdot n \geq 0, \\
\lambda_{n,E} & \text{else}.
\end{cases}
\]

Hence depending on the value of \(\beta \in \{0, 1\}\), we might use central differencing or upwinding of the mobility for internal interfaces.
Phase conservation equation

\[(\Phi \partial_t s_{n,h,z}) + c_h(p_{w,h};s_{n,h,z}) + d_h(s_{n,h,z}) = r_h(z) \quad \forall z \in D_r(T_h), \forall t \in J.\]

The diffusion term \(-\nabla \cdot (\lambda_n K \nabla p_c)\) is discretized by a bilinear form similar to that of (29).

\[
d_h(s_{n,h,z}) = \sum_{E \in T_h} \int_E \lambda_n K \nabla p_{c,h} \cdot \nabla z - \sum_{e \in \Gamma^h_D} \int_e \{\lambda_n K \nabla p_{c,h}\} \omega \cdot n_e [z]
+ \epsilon \sum_{e \in \Gamma^h_D} \int_e \{\lambda_n K \nabla z\} \omega \cdot n_e [s_{n,h}] + \sum_{e \in \Gamma^h_D} \gamma_e^s \int_e [s_{n,h}] [z]. \quad (29)
\]
Phase conservation equation

\[(\Phi \partial_t s_{n,h}, z) + c_h(p_{w,h}; s_{n,h}, z) + d_h(s_{n,h}, z) = r_h(z) \quad \forall z \in D_r(T_h), \forall t \in J.\]

The diffusion term \(-\nabla \cdot (\lambda_n \nabla p_c)\) is discretized by a bilinear form similar to that of (29).

\[d_h(s_{n,h}, z) = \sum_{E \in T_h} \int_E \lambda_n K \nabla p_c, h \cdot \nabla z - \sum_{e \in \Gamma^h \cup \Gamma^h_D} \int_e \{\lambda_n \nabla p_c, h\}_w \cdot n_e[z] + \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_D} \int_e \lambda_n \nabla z \cdot n_e \omega \cdot n_e p_D + \epsilon \sum_{e \in \Gamma^h \cup \Gamma^h_D} \gamma_e^s \int_e [s_{n,h}] [z].\]  

(29)

The right hand side \(r_h\) includes the Neumman and Dirichlet boundary condition and the nonwetting source term.

\[r_h(z) = \int_\Omega q_n z - \sum_{e \in \Gamma^h_N} \int_e J_n z + \epsilon \sum_{e \in \Gamma^h_D} \int_e \lambda_n \nabla z \cdot n_e p_D + \epsilon \sum_{e \in \Gamma^h_D} \int_e \lambda_n K \nabla z \cdot n_e p_c(s_D) + \sum_{e \in \Gamma^h_D} \gamma_e^s \int_e s_D z, \quad \forall z \in D_r(T_h).\]  

(30)

where:

\[\gamma_e^s = C_s \frac{p(p+d-1)e}{\min(|E_+|, |E_-|)}, \quad c_s \geq 0.\]  

(31)
Application of Backward Euler time discretization and Interior Penalty DG for space discretization to the system of PDEs for the coupled equations (19) - (19) yields:

\[
\mathcal{B}_h(p_{w,h}^{i+1}, s_{n,h}^{i+1}, v) = l_h(v), \quad \forall v \in \mathcal{D}_p(\mathcal{T}_h), \tag{32}
\]

\[
(\Phi \frac{s_{n,h}^{i+1} - s_{n,h}^i}{\Delta t}, z) + c_h(p_{w,h}^{i+1}; s_{n,h}^{i+1}, z) + d_h(s_{n,h}^{i+1}, z) = r_h(z), \quad \forall z \in \mathcal{D}_s(\mathcal{T}_h), \tag{33}
\]

\[
(s_{n,h}^0, \phi) = (s_{n}^0, \phi), \quad \forall \phi \in \mathcal{D}_s(\mathcal{T}_h). \tag{34}
\]
A posteriori estimator & adaptivity state of work

- Karakashian & Pascal (2003)
  - A posteriori error estimation for pure diffusion

  - Stationary convection-diffusion equations

- Cangiani, Metcafe & Georgoulis 2013
  - Non-stationary convection-diffusion problems

- Sun & Wheeler (2005)
  - Coupled diffusion and advection dominated transport

- Vohralik & Wheeler (2013)
  - General abstract framework, a posteriori estimates for immiscible incompressible two-phase flows in porous media.
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  - A posteriori error estimation for pure diffusion
  - Stationary convection-diffusion equations
- Cangiani, Metcafe & Georgoulis 2013
  - non-stationary convection-diffusion problems
  - $L^2(H^1) + L^\infty(L^2)$ type norm
- Sun & Wheeler (2005)
  - coupled diffusion and advection dominated transport
- Vohralik & Wheeler (2013)
  - general abstract framework,
  - a posteriori estimates for immiscible incompressible two-phase flows in porous media.
Challenge: Develop efficient implementation with DG on a sustainable framework.

Software framework

- Built on top of Dune-Fem
  - user-friendly implementation,
  - profit from many features,
  - DG discretization spaces,
  - efficient solvers and grids,
  - support for parallelization and adaptivity.

Dune-Fem-twophaseDG

Dune-Fem

Dune-Grid

Dune-localfunctions

Dune-Istl

Dune-Common

Dune-Fem

Dune-Grid

Dune-localfunctions

Dune-Istl

Dune-Common
2d infiltration problem

Abbildung: Geometry and boundary conditions for the DNAPL infiltration problem

<table>
<thead>
<tr>
<th></th>
<th>$\Omega_{lens}$</th>
<th>$\Omega \setminus \Omega_{lens}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi$ [-]</td>
<td>0.39</td>
<td>0.40</td>
</tr>
<tr>
<td>$k$ [m$^2$]</td>
<td>$6.64 \times 10^{-16}$</td>
<td>$6.64 \times 10^{-11}$</td>
</tr>
<tr>
<td>$S_{wr}$ [-]</td>
<td>0.1</td>
<td>0.12</td>
</tr>
<tr>
<td>$S_{nr}$ [-]</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$\theta$ [-]</td>
<td>2.0</td>
<td>2.70</td>
</tr>
<tr>
<td>$p_d$ [Pa]</td>
<td>5000</td>
<td>755</td>
</tr>
</tbody>
</table>

Tabelle: Parameters

<table>
<thead>
<tr>
<th></th>
<th>$\Gamma_{IN}$</th>
<th>$J_n = -0.075, J_w = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_N$</td>
<td>$J_n = 0.00, J_w = 0.00$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_S$</td>
<td>$J_w = 0, J_n = 0.00$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_E \cup \Gamma_W$</td>
<td>$p_w = (0.65 - y) \cdot 9810, s_n = 0$</td>
<td></td>
</tr>
</tbody>
</table>

Tabelle: Boundary conditions
2d infiltration problem

Abbildung: DNAPL saturation distribution after 2000 s (left), leaf grid (right).
Polynomial order $p = 3$.

Abbildung: Comparison of non-wetting-phase saturation at $T=2000$ s. Left, profile along the line $((0,0);(0.45,0.65))$. Right, profile along the line $x=0.45$ m.
3d infiltration problem

Abbildung: Geometry of the domain for the 3d DNAPL infiltration problem

<table>
<thead>
<tr>
<th></th>
<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
<th>$\Omega \setminus \Omega_1 \cap \Omega \setminus \Omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi$ [-]</td>
<td>0.39</td>
<td>0.39</td>
<td>0.40</td>
</tr>
<tr>
<td>$k$ [m$^2$]</td>
<td>$6.64 \times 10^{-16}$</td>
<td>$6.64 \times 10^{-15}$</td>
<td>$6.64 \times 10^{-11}$</td>
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<tr>
<td>$S_{wr}$ [-]</td>
<td>0.1</td>
<td>0.1</td>
<td>0.12</td>
</tr>
<tr>
<td>$S_{nr}$ [-]</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$\theta$ [-]</td>
<td>2.0</td>
<td>2.0</td>
<td>2.70</td>
</tr>
<tr>
<td>$p_d$ [Pa]</td>
<td>5000</td>
<td>5000</td>
<td>755</td>
</tr>
</tbody>
</table>

Abbildung: 3d problem parameters
3d infiltration problem

Abbildung: Contour plot of saturation distribution after 3600 s of injection with 0.25 Kg s$^{-1}$m$^{-2}$ of DNAPL in a depth of 1 m (left column), leaf grid (center column) and saturation profile along the line ((1,0.45,0);(0.45,0.45,1))(right column).
3d infiltration problem

Abbildung: 3d-Problem: Saturation distribution after 3000 s of injection with 0.25 kg s$^{-1}$ m$^{-2}$ of DNAPL. Unstructured mesh. Polynomial order $p = 1$. 
Test case with analytical solution

We consider a system of partial differential equations with known exact solution.

**Problem**

Considering $\Omega = (0,1)^2$ and $J = (0,T)$, find $(p,s)$ such that

\[
- \nabla \cdot (\lambda(s)K \nabla p) = 0 \quad \text{in } \Omega \times J \tag{35}
\]

\[
\phi \frac{\partial s}{\partial t} - \nabla \cdot (-\epsilon \nabla s + f(s)\lambda(s)K \nabla p) = q \quad \text{in } \Omega \times J \tag{36}
\]

with $\lambda(s) = (0.5 - 0.2s)^{-1}$, $\epsilon = 0.01$, $f(s) = s$, where $q = 2\pi \epsilon \sin(\pi(x_1 + x_2 - 2t))$

**Boundary & Initial conditions**

\[
p(x,y,t) = \frac{0.2}{\pi} \cos(\pi(x + y - 2t)) - 0.5(x+y), \quad s(x,y,t) = \sin(\pi(x + y - 2t)) \quad \forall (x,y,t) \in \partial\Omega \times J
\]

\[
p(x,y,t) = \frac{0.2}{\pi} \cos(\pi(x + y - 2t)) - 0.5(x+y), \quad s(x,y,t) = \sin(\pi(x + y - 2t)) \quad \forall (x,y,t) \in \Omega
\]
Test case with analytical solution

<table>
<thead>
<tr>
<th>AvgNb\text{dofs}</th>
<th>MaxNb\text{dofs}</th>
<th>|p − p_h|_{L^2(\Omega)}</th>
<th>|s − s_h|_{L^2(\Omega)}</th>
</tr>
</thead>
<tbody>
<tr>
<td>665.15</td>
<td>792</td>
<td>1.110^{−2}</td>
<td>4.7 10^{−2}</td>
</tr>
<tr>
<td>1944.47</td>
<td>2570</td>
<td>4.9410^{−3}</td>
<td>2.87 10^{−2}</td>
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<tr>
<td>3483</td>
<td>4624</td>
<td>3.7110^{−3}</td>
<td>2.62 10^{−2}</td>
</tr>
</tbody>
</table>

Tabelle: TYPEPARAM algorithm (200 time steps, T=0.2 [s], Newton tol 1e-6, residual tol 1e-7)

<table>
<thead>
<tr>
<th>AvgNb\text{dofs}</th>
<th>MaxNb\text{dofs}</th>
<th>|p − p_h|_{L^2(\Omega)}</th>
<th>|s − s_h|_{L^2(\Omega)}</th>
</tr>
</thead>
<tbody>
<tr>
<td>722</td>
<td>960</td>
<td>1.410^{−2}</td>
<td>4.64 10^{−2}</td>
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<tr>
<td>2583.1</td>
<td>3108</td>
<td>4.85410^{−3}</td>
<td>3.29 10^{−2}</td>
</tr>
</tbody>
</table>

Tabelle: ERNDOVO algorithm (200 time steps, T=0.2 [s], Newton tol 1e-6, residual tol 1e-7)

Abbildung: Saturation profiles at $T = 0.2$

Abbildung: Polynomial degrees at $T = 0.2$
h-p test Case

In summary

- Fully-coupled discontinuous Galerkin (DG) method for incompressible two-phase flow with discontinuous capillary pressure,
  - no post-processing of the DG velocity field in contrast to results for decoupled schemes
- Interior Penalty DG formulation,
  - Weighted averages,
- $h$, $p$ and $h$-$p$ adaptivity,
- Higher order polynomials up to piecewise cubics are implemented.
Thank you for your interest!