Monotone nonlinear Finite Volume Scheme for Flow in Porous Media

OPM Meeting, June 1st, 2016

Martin Schneider, Bernd Flemisch, Rainer Helmig
University of Stuttgart

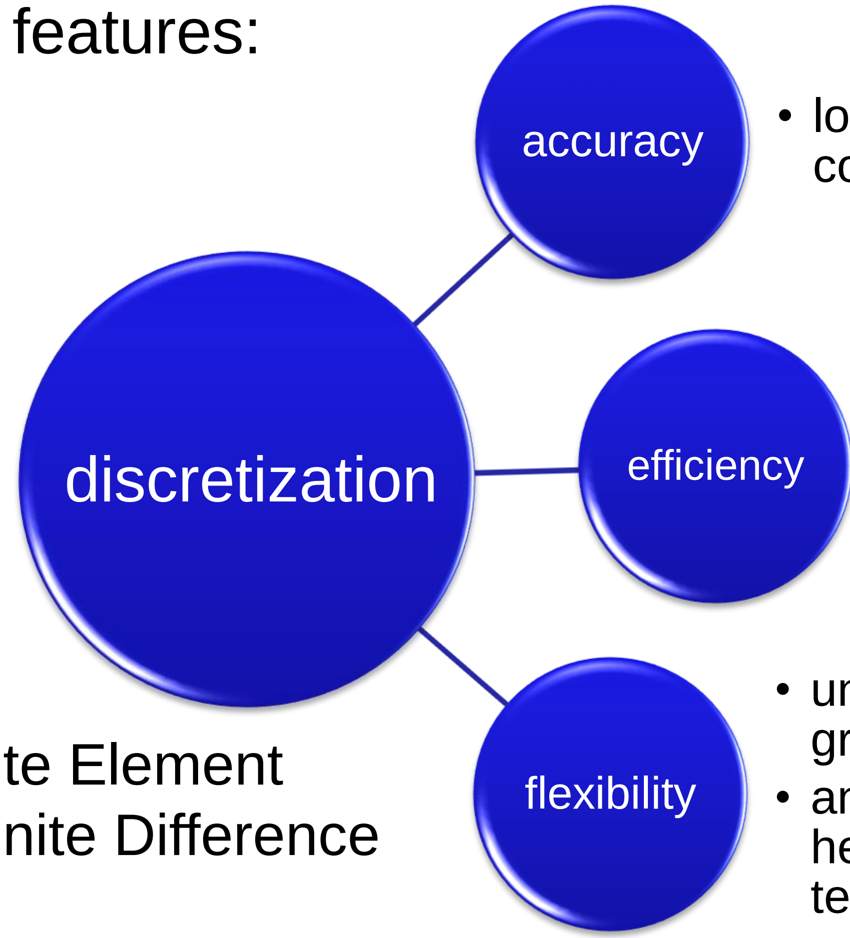
Kirill Terekhov, Hamdi Tchelepi
Stanford University

Overview DuMu^X

- **DuMu^x**: DUNE for Multi-{Phase, Component, Scale, Physics, ...} Flow and Transport in Porous Media
- Development started 2007
- Based on **DUNE**
- Current release: 2.9 (March 2016, git repository)
- Recently started to use opm-grid (corner-point grids), “straight forward” because of the dune grid interface
- Interesting for us: using other opm modules like upscaling …

Why nonlinear Finite Volume Methods?

Desirable features:



- locally mass conservative

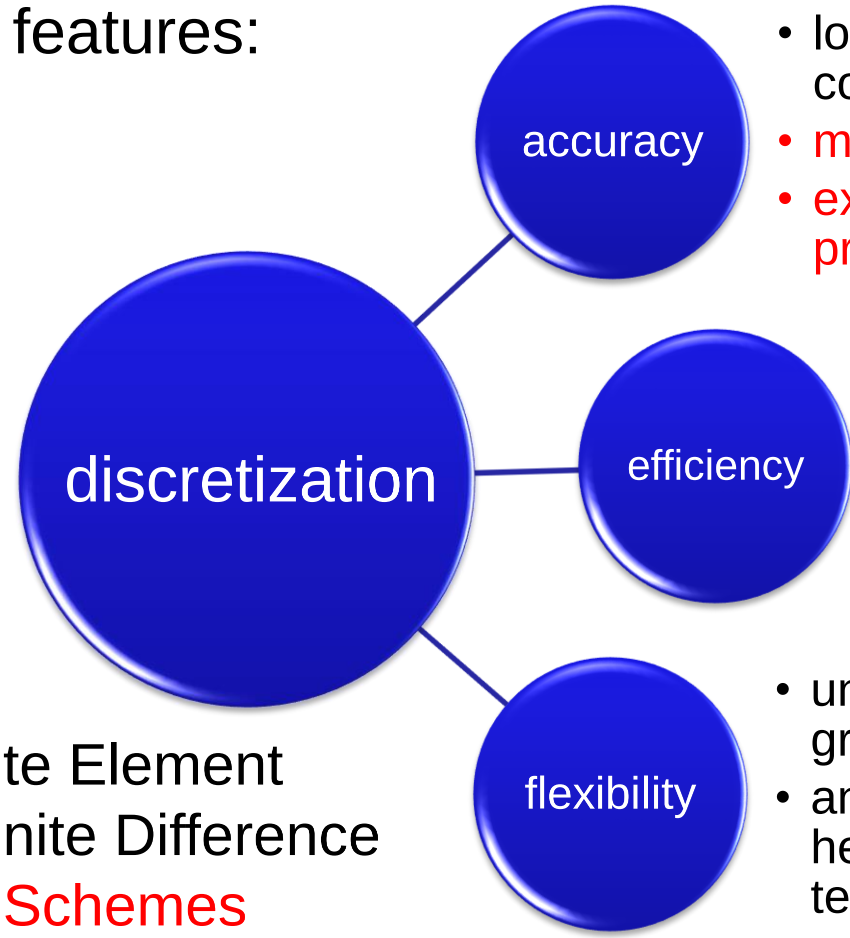
- sparse matrices

- unstructured grids
- anisotropic, heterogeneous tensors

- MPFA
- Mixed Finite Element
- Mimetic Finite Difference

Why nonlinear Finite Volume Methods?

Desirable features:



- locally mass conservative
- monotone
- extremum principles

- sparse matrices

- unstructured grids
- anisotropic, heterogeneous tensors

- MPFA
- Mixed Finite Element
- Mimetic Finite Difference
- Nonlinear Schemes

Idea of flux calculation

Flux Approximation: (elliptic equation)

$$-\nabla \cdot (\mathbf{K}\nabla p) = q \quad \longrightarrow \quad -\int_{\partial V_i} (\mathbf{K}\nabla p) \cdot \mathbf{n} \, dS = \int_{V_i} q \, dV$$

$$\mathbf{K} = \mathbf{K}^T, \quad \mathbf{d} := \mathbf{K} \cdot \mathbf{n} \quad \longrightarrow \quad -\sum_{\sigma} \int_{\sigma} \nabla p \cdot \mathbf{d} \, dS = \int_{V_i} q \, dV$$

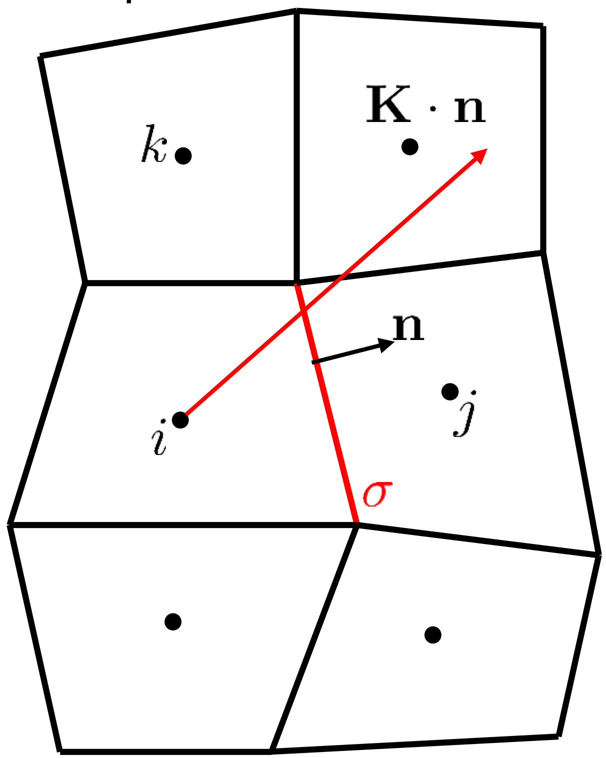
For each cell face $\sigma \subset \partial V_i$ approximate the fluxes

$$f_\sigma = -|\sigma| \nabla p \cdot \mathbf{d}, \; \mathbf{d} := \mathbf{K} \cdot \mathbf{n}$$

Idea of flux calculation for homogeneous permeability tensor

$$f_1 = -|\sigma|\nabla p \cdot \mathbf{d}, \; \mathbf{d} := \mathbf{K} \cdot \mathbf{n}$$

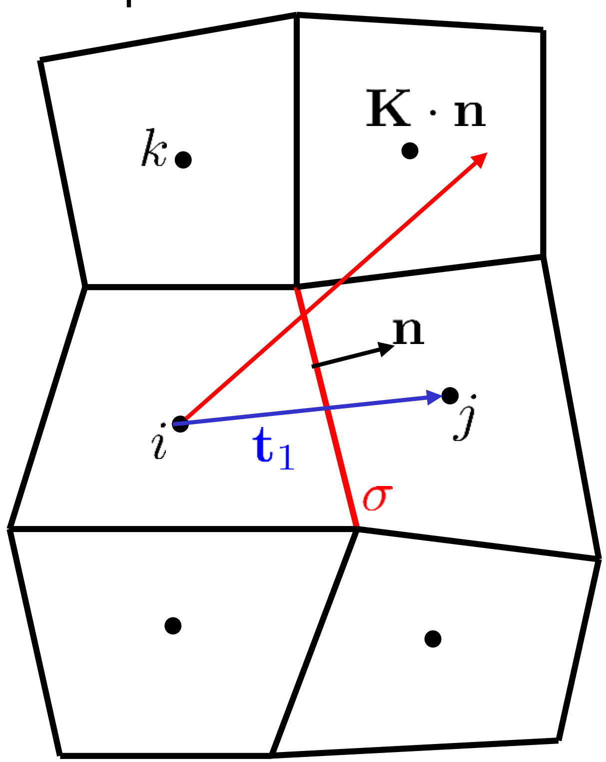
conormal decomposition:



Idea of flux calculation for homogeneous permeability tensor

$$f_1 = -|\sigma| \left(\alpha_1 (p_j - p_i) + \qquad\qquad\qquad\qquad \right)$$

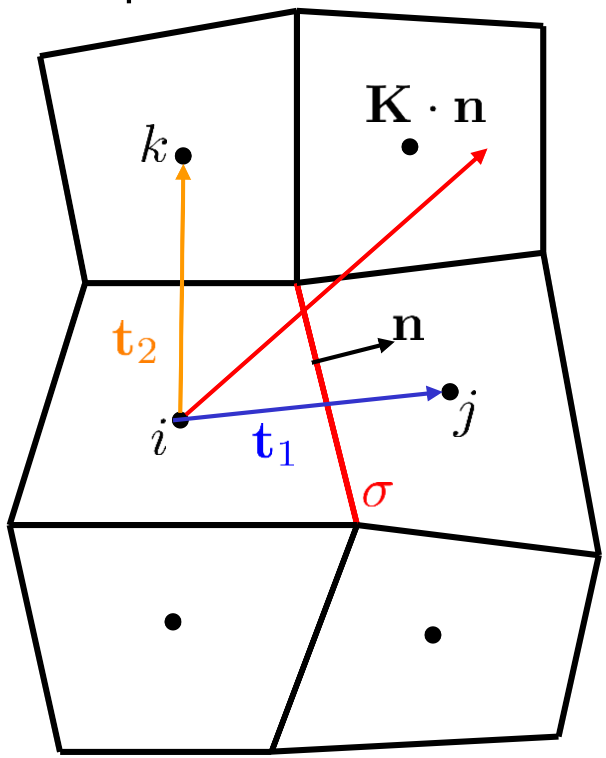
conormal decomposition:



Idea of flux calculation for homogeneous permeability tensor

$$f_1 = -|\sigma| \left(\alpha_1 (p_j - p_i) + \alpha_2 (p_k - p_i) \right)$$

conormal decomposition:

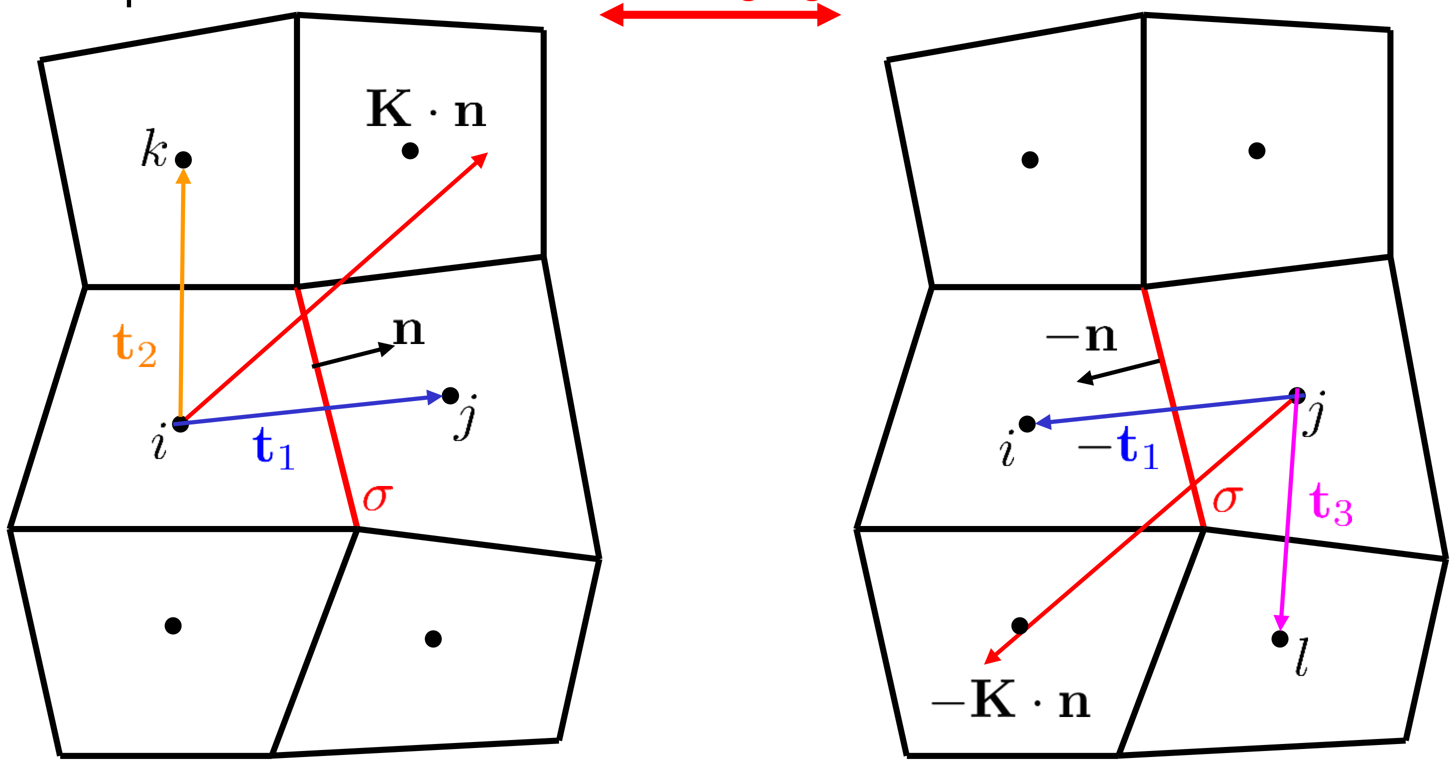


Idea of flux calculation for homogeneous permeability tensor

$$f_1 = -|\sigma| \left(\alpha_1 (p_j - p_i) + \alpha_2 (p_k - p_i) \right) \qquad f_2 = -|\sigma| \left(\beta_1 (p_i - p_j) + \beta_2 (p_l - p_j) \right)$$

conormal decomposition: $\boxed{\alpha, \beta \geq 0}$

averaging



Averaged Flux Approximation

$$\rightarrow \quad f_\sigma := \mu_1 f_1 - \mu_2 f_2, \qquad \mu_1 + \mu_2 = 1, \quad 0 \leq \mu_1, \mu_2 \leq 1$$

$$\begin{aligned} = \quad & |\sigma| \left(\mu_1 \left(\alpha_1 + \alpha_2\right) + \mu_2 \beta_1\right) p_i \\ - & |\sigma| \left(\mu_2 \left(\beta_1 + \beta_2\right) + \mu_1 \alpha_1\right) p_j \\ - & |\sigma| \left(\mu_1 \alpha_2 p_k - \mu_2 \beta_2 p_l\right) \end{aligned}$$

Possible choice: $\mu_1 = \mu_2 = 0.5$

Averaged Multi-Point Flux Approximation: AvgMPFA

Averaged Flux Approximation

$$\rightarrow \quad f_\sigma := \mu_1 f_1 - \mu_2 f_2, \qquad \mu_1 + \mu_2 = 1, \quad 0 \leq \mu_1, \mu_2 \leq 1$$

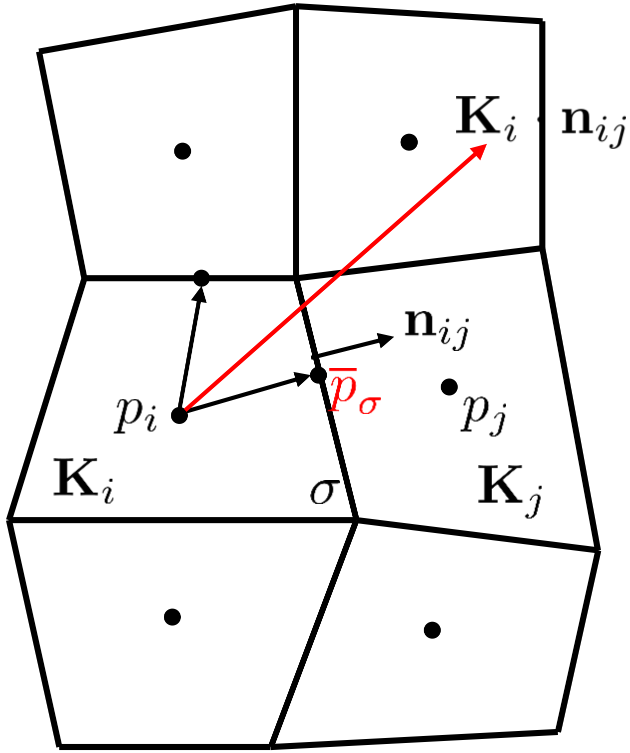
$$\begin{aligned} = \quad & |\sigma| \left(\mu_1 \left(\alpha_1 + \alpha_2\right) + \mu_2 \beta_1\right) p_i \\ & - |\sigma| \left(\mu_2 \left(\beta_1 + \beta_2\right) + \mu_1 \alpha_1\right) p_j \\ & - |\sigma| \underbrace{\left(\mu_1 \alpha_2 p_k - \mu_2 \beta_2 p_l\right)}_{\overset{!}{=} 0} \end{aligned}$$

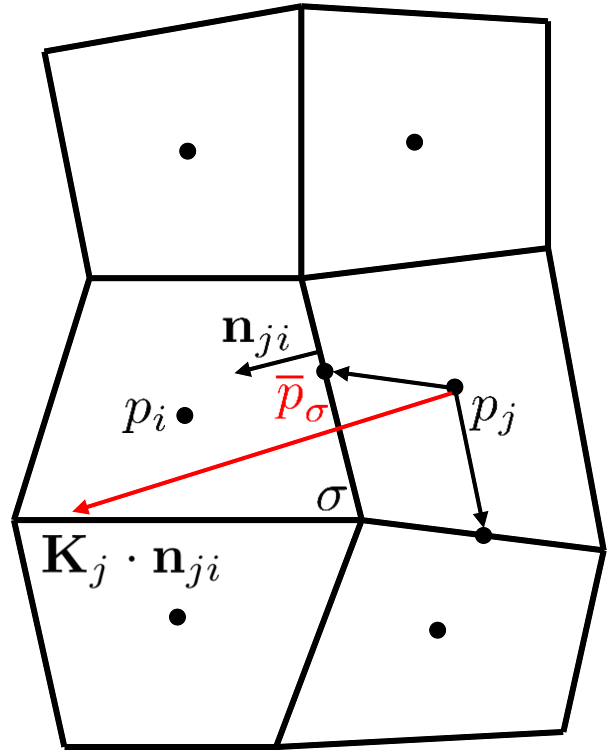
$$\rightarrow f_\sigma = t_1(p_k, p_l) p_i - t_2(p_k, p_l) p_j$$

Nonlinear Flux Approximation: NLTPFA

- Le Potier, C. (2005). „*Finite volume monotone scheme for highly anisotropic diffusion operators on unstructured triangular meshes.*“
- Danilov, A. A., & Vassilevski, Y. V. (2009). “*A monotone nonlinear finite volume method for diffusion equations on conformal polyhedral meshes.*”

Heterogeneous permeability tensor



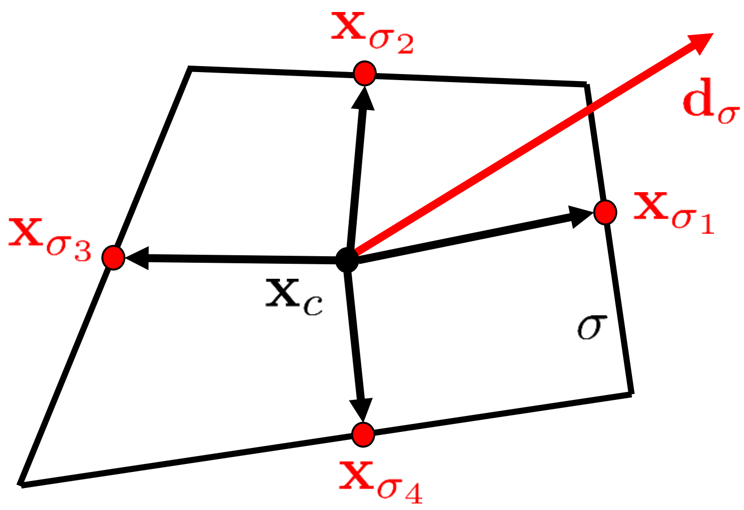


Harmonic Averaging Point: $\overline{p}_\sigma = \frac{c_i p_i + c_j p_j}{c_i + c_j}.$

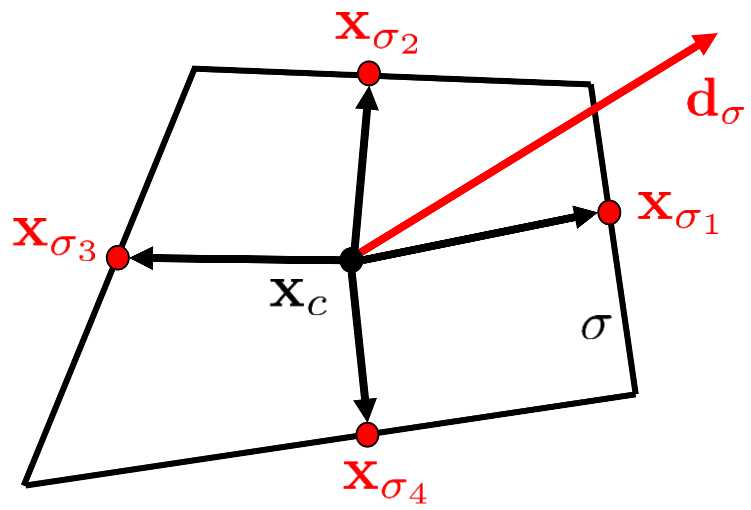
Agélas et al., "*A nine-point finite volume scheme for the simulation of diffusion in heterogeneous media.*"

$$\overline{\mathbf{x}}_\sigma = \frac{c_i \mathbf{x}_i + c_j \mathbf{x}_j + (\mathbf{K}_i - \mathbf{K}_j)\mathbf{n}_{ij}}{c_i + c_j}, \quad c_i = \frac{\mathbf{n}_{ij} \cdot \mathbf{K}_i \cdot \mathbf{n}_{ij}}{\mathrm{dist}(\mathbf{x}_i, \sigma)}, \quad c_j = \frac{\mathbf{n}_{ij} \cdot \mathbf{K}_j \cdot \mathbf{n}_{ij}}{\mathrm{dist}(\mathbf{x}_j, \sigma)}$$

Generalization of conormal decomposition



Generalization of conormal decomposition



$$\min_{\alpha \in \mathbb{R}^{NF}} F(\alpha)$$

$$\text{subject to} \quad \mathbf{d}_\sigma = \mathbb{A}\alpha$$

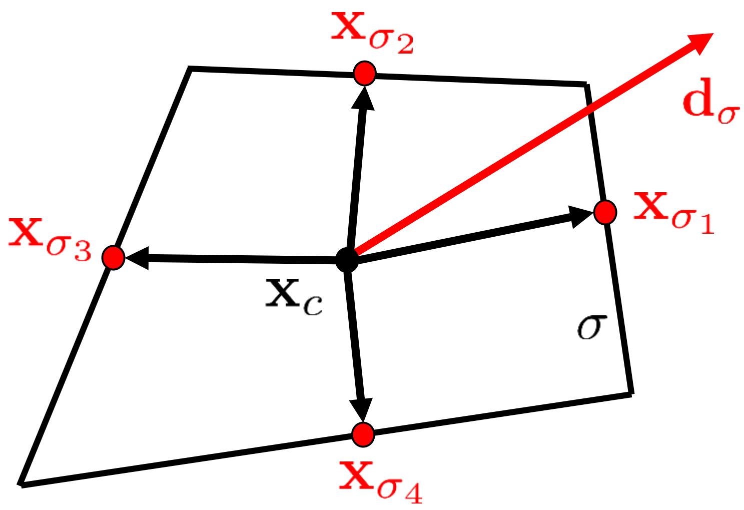
$$\sum \alpha_i \geq c, \quad \alpha_i \geq 0$$

$$\mathbb{A} = \left(\mathbf{x}_{\sigma_1} - \mathbf{x}_c, \cdots, \mathbf{x}_{\sigma_{NF}} - \mathbf{x}_c\right)$$

example:

$$F(\alpha) = \sum \alpha_i$$

Generalization of conormal decomposition



$$\min_{\gamma \geq 0,\ \alpha \in \mathbb{R}^{NF}} \kappa\gamma + F(\alpha)$$

$$\text{subject to} \quad \mathbf{d}_\sigma = \mathbb{A}\alpha$$

$$\sum \alpha_i \geq c, \quad \alpha_i \geq -\gamma$$

$$\mathbb{A} = \left(\mathbf{x}_{\sigma_1} - \mathbf{x}_c, \cdots, \mathbf{x}_{\sigma_{NF}} - \mathbf{x}_c\right)$$

example:

$$F(\alpha) = \sum \alpha_i$$

$$\kappa >> F(\alpha)$$

→ If $\gamma > 0$ then there exists no conormal decomposition with only positive coefficients

Modification of nonlinear TPFA

$$f_\sigma = \quad t_1(\mu_1,\mu_2)p_i - t_2(\mu_1,\mu_2)p_j \underbrace{-\mu_1\lambda_1 + \mu_2\lambda_2}_{=:R_\sigma \overset{!}{=} 0}$$

$$\lambda_1 := \sum \alpha_{k,j} p_j, \qquad \lambda_2 := \sum \alpha_{l,j} p_j.$$

$\rightarrow$ $R_\sigma = 0, \quad \text{if } \lambda_1\lambda_2 > 0$

$$\text{if } \lambda_1\lambda_2 < 0: \qquad f_\sigma = \left(t_1(\mu_1,\mu_2) + \frac{|R_\sigma| + R_\sigma}{2(p_i + \varepsilon)} \right) p_i - \left(t_2(\mu_1,\mu_2) + \frac{|R_\sigma| - R_\sigma}{2(p_j + \varepsilon)} \right) p_j$$

Literature overview NLTPFA

Existing results in literature:

- proof of monotonicity
- convergence behavior for elliptic equation (numerically)
  (second order for pressure, first order for flux)

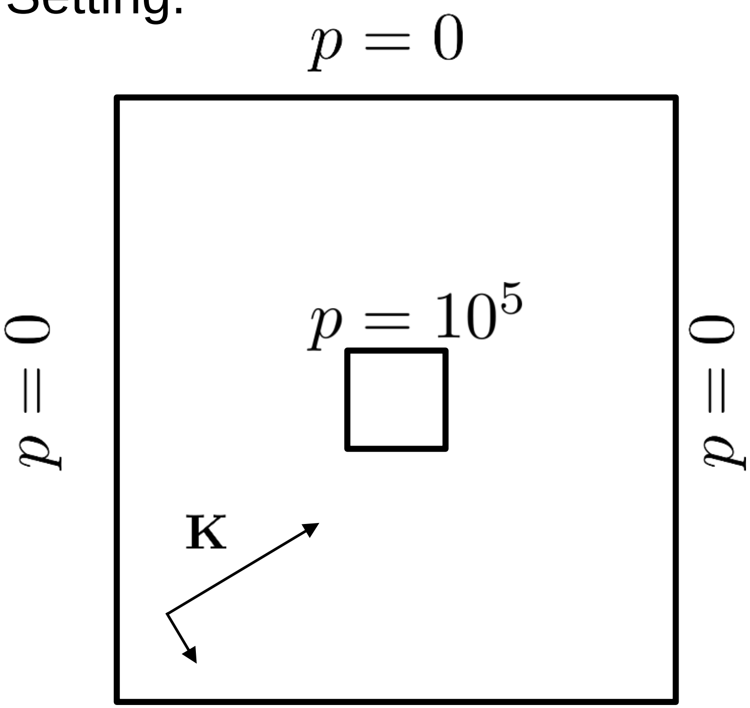
Focus of this talk:

- comparison with commonly used linear schemes
- influence of additional nonlinearity due to nonlinear flux approximation
- challenging grids
- complex physics

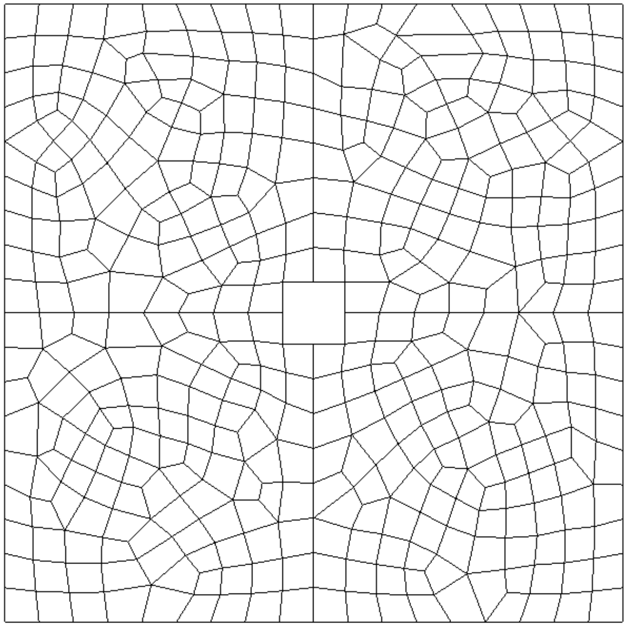
Monotonicity

$$-\nabla \cdot (\mathbf{K}\nabla p) = 0, \quad \Omega = [0,1] \times [0,1],$$

Setting:



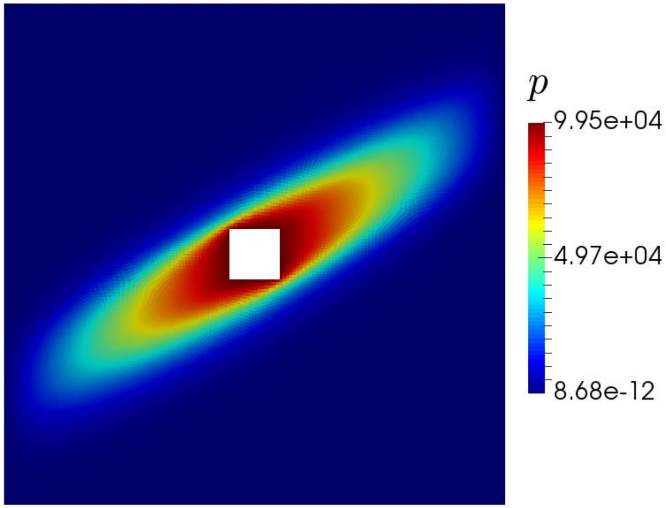
unstructured grid



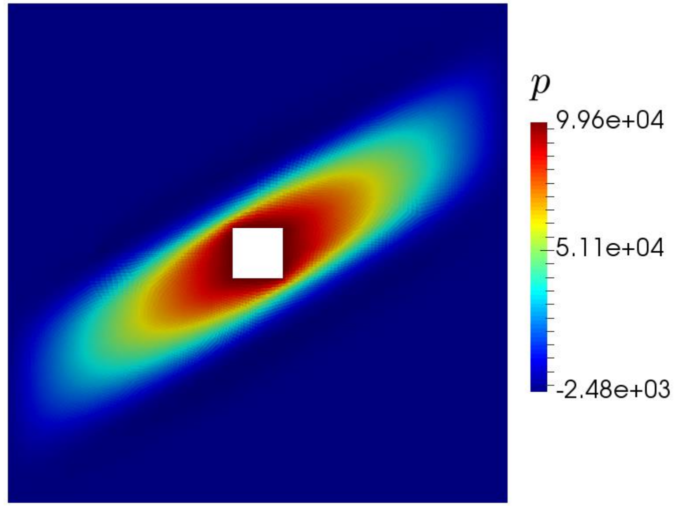
$$\mathbf{K} = \mathbf{R}(\tfrac{\pi}{6}) \begin{pmatrix} 1000 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}(\tfrac{\pi}{6})^{-1}$$

Monotonicity

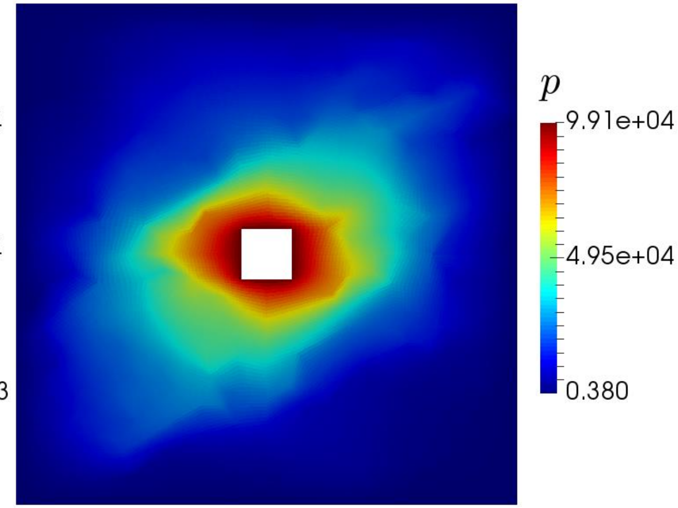
NLTPFA (nnz 192544)



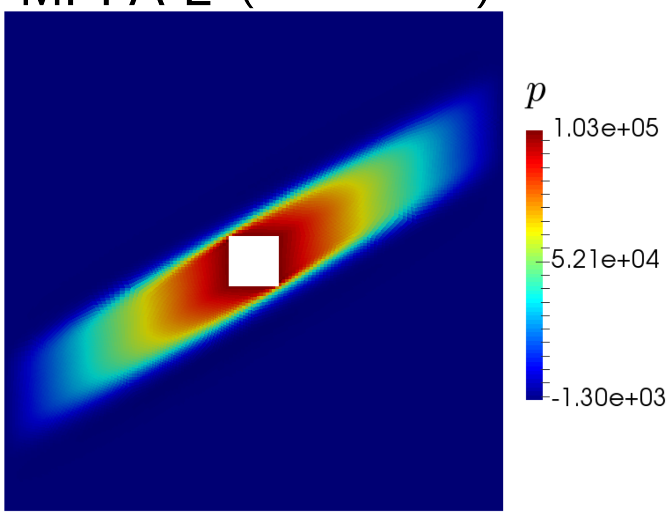
AvgMPFA (nnz 192544)



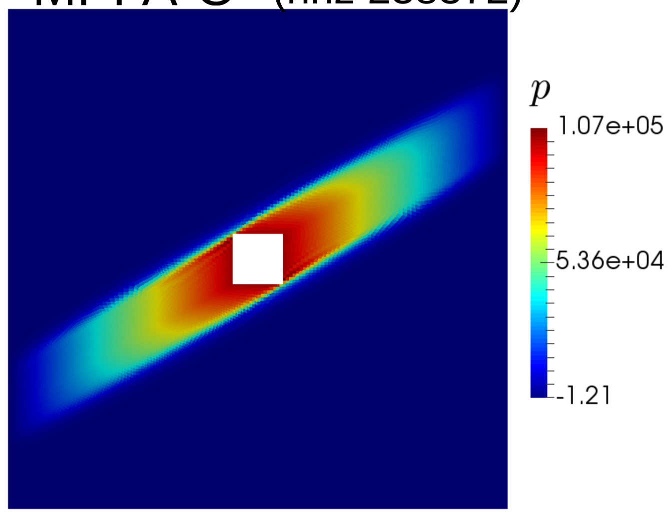
TPFA (nnz 132800)



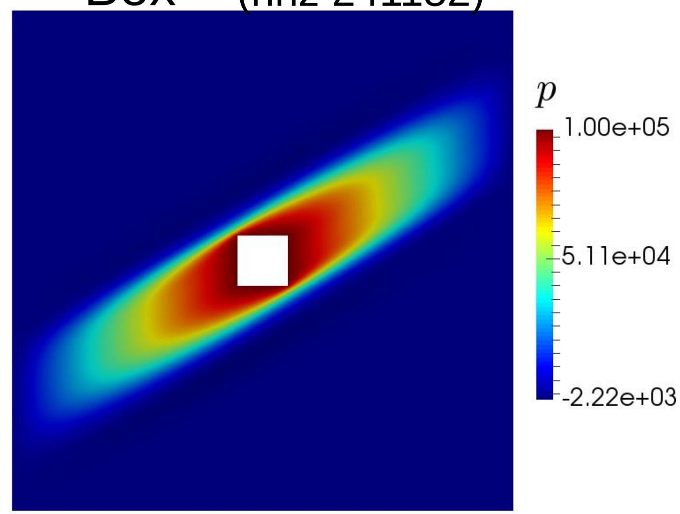
MPFA-L (nnz 185883)



MPFA-O (nnz 238372)

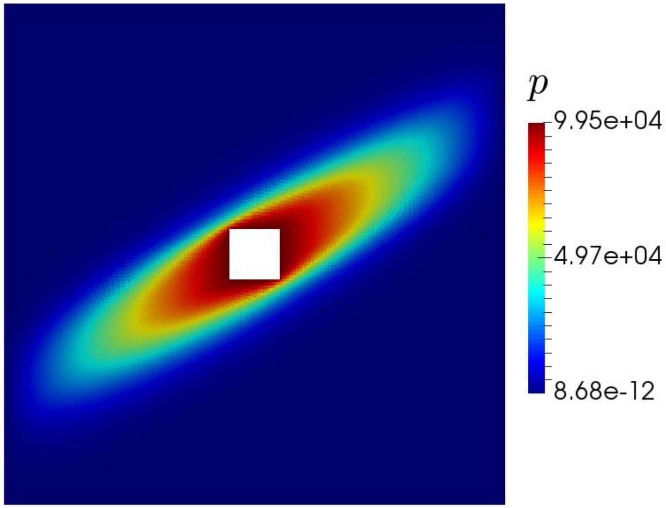


Box (nnz 241152)

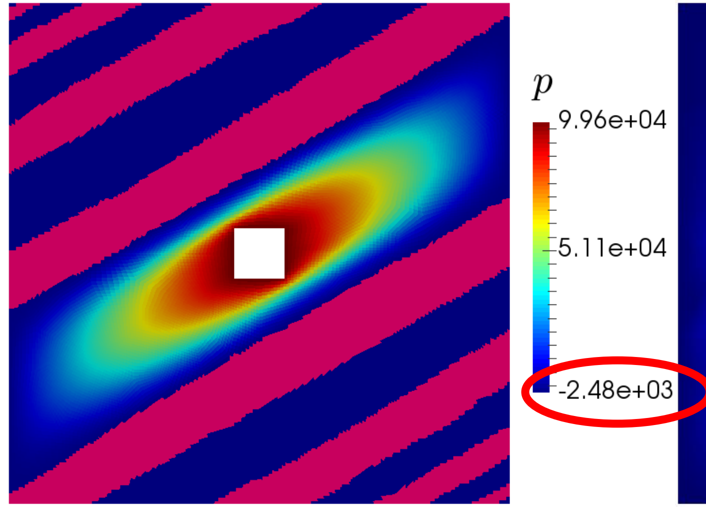


Monotonicity

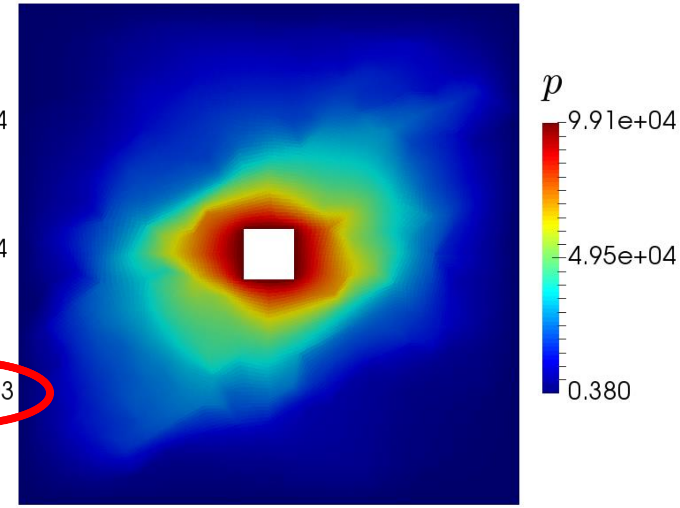
NLTPFA



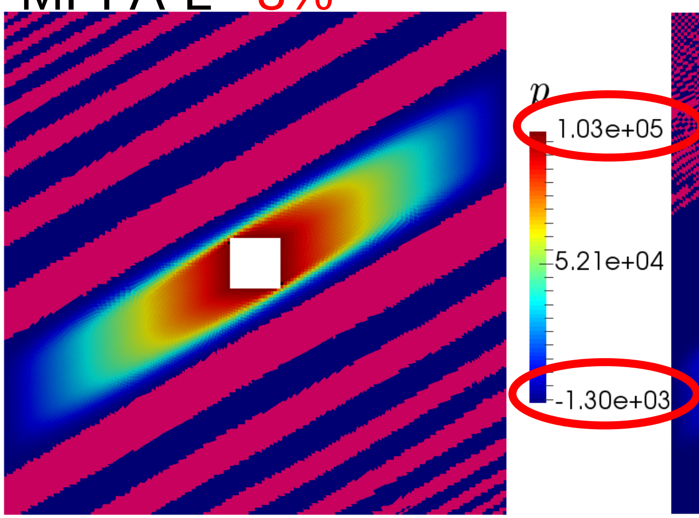
AvgMPFA ~2.5%



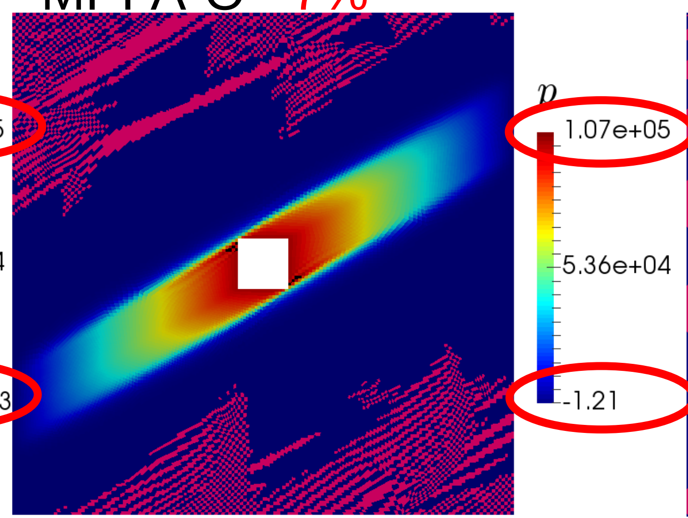
TPFA



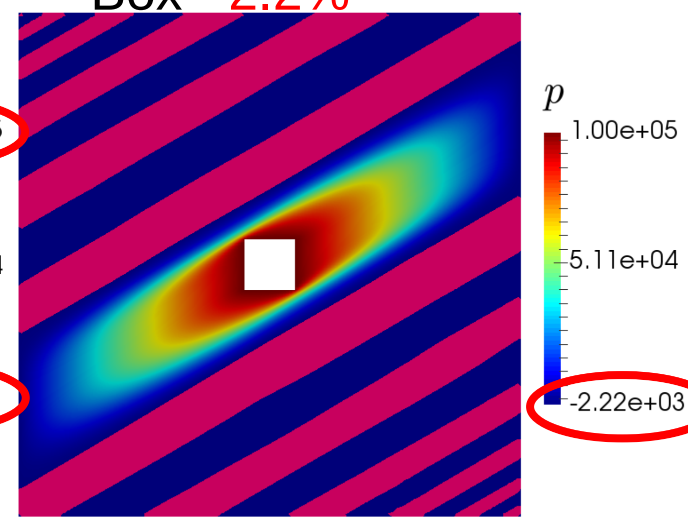
MPFA-L ~3%



MPFA-O ~7%



Box ~2.2%

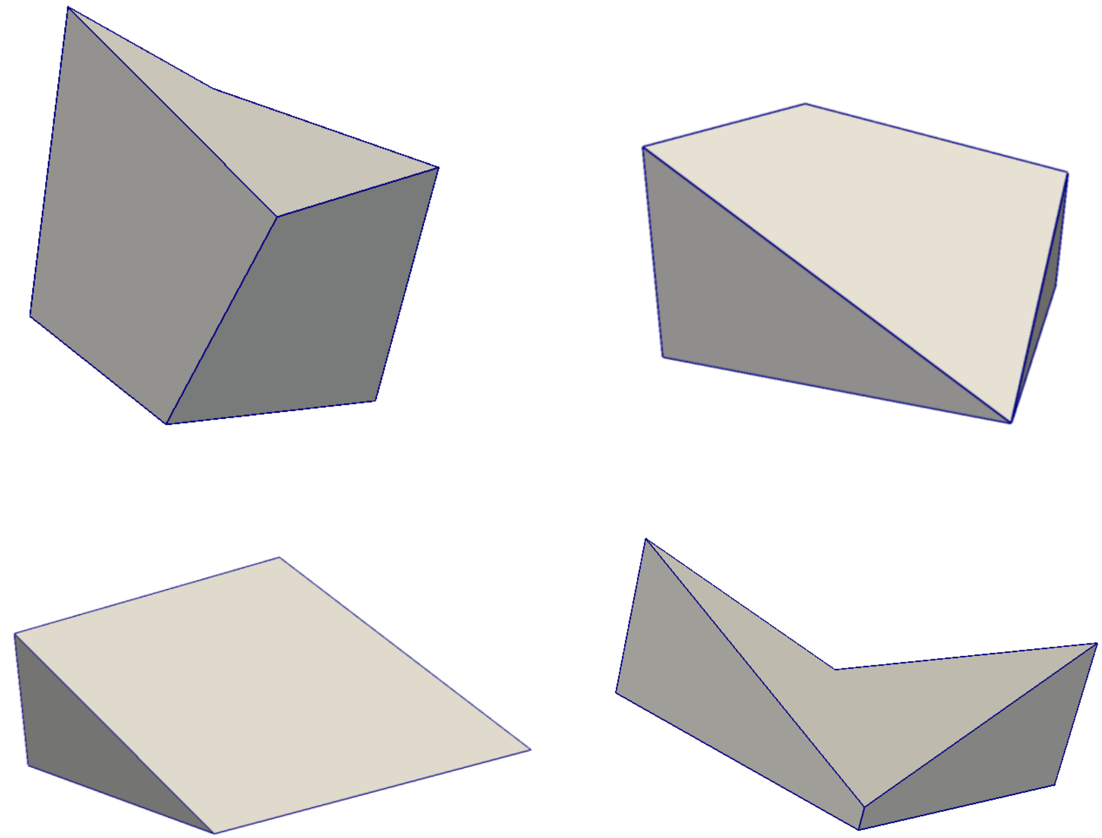


Properties of NLTPFA

- second order accuracy for pressure, first order for flux (numerical results)
- monotonicity

Challenging grids (corner-point grids)

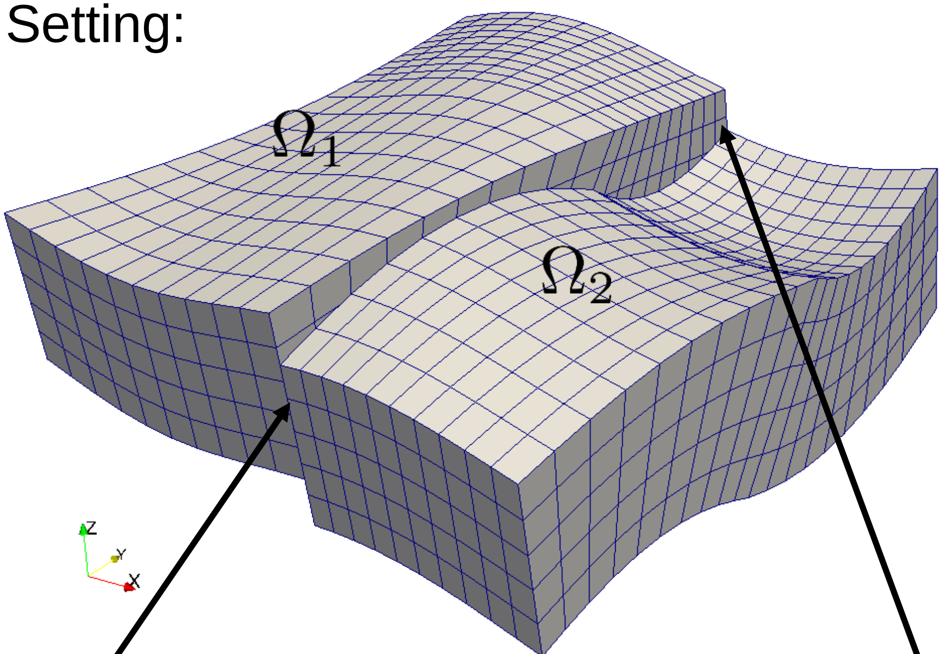
- curved faces
- degenerated points
- degenerated faces
- non-convex cells
- centroid outside of cell
- non-matching grids



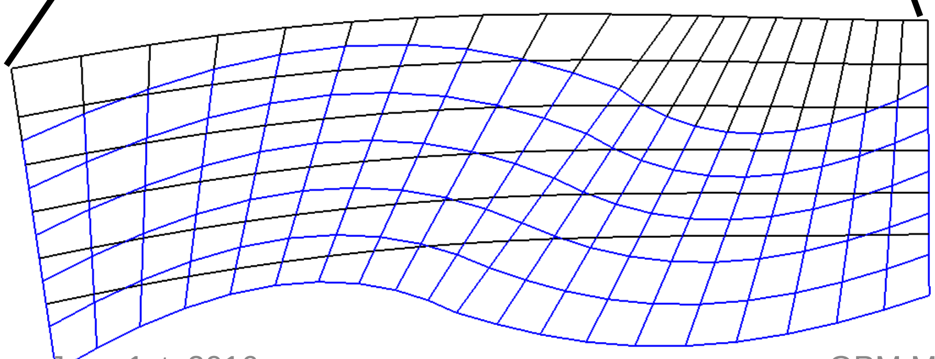
opm-grid

Example: non-matching grid

Setting:



generated with MRST



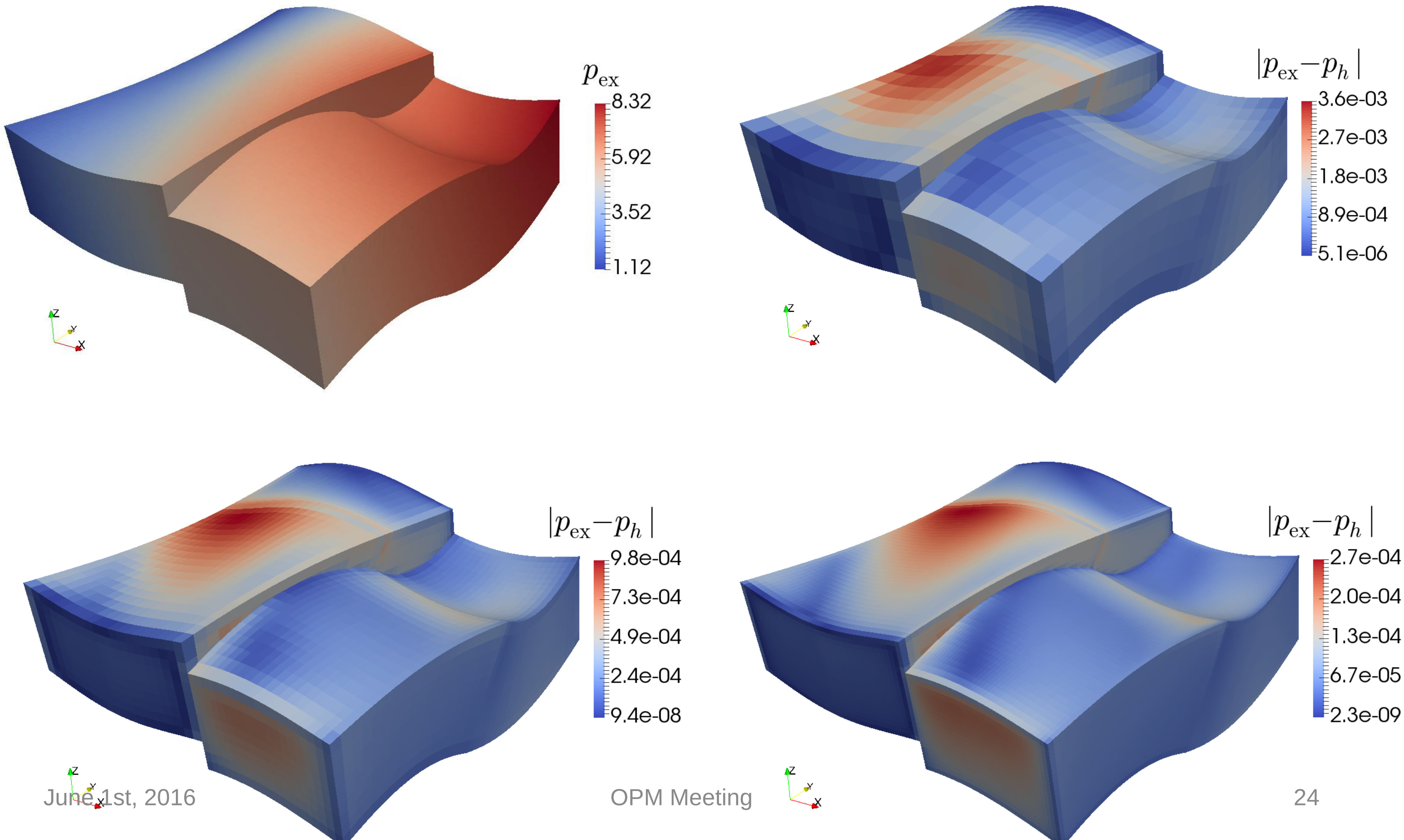
$$\mathbf{K}_1 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{K}_2 = \begin{pmatrix} 10 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

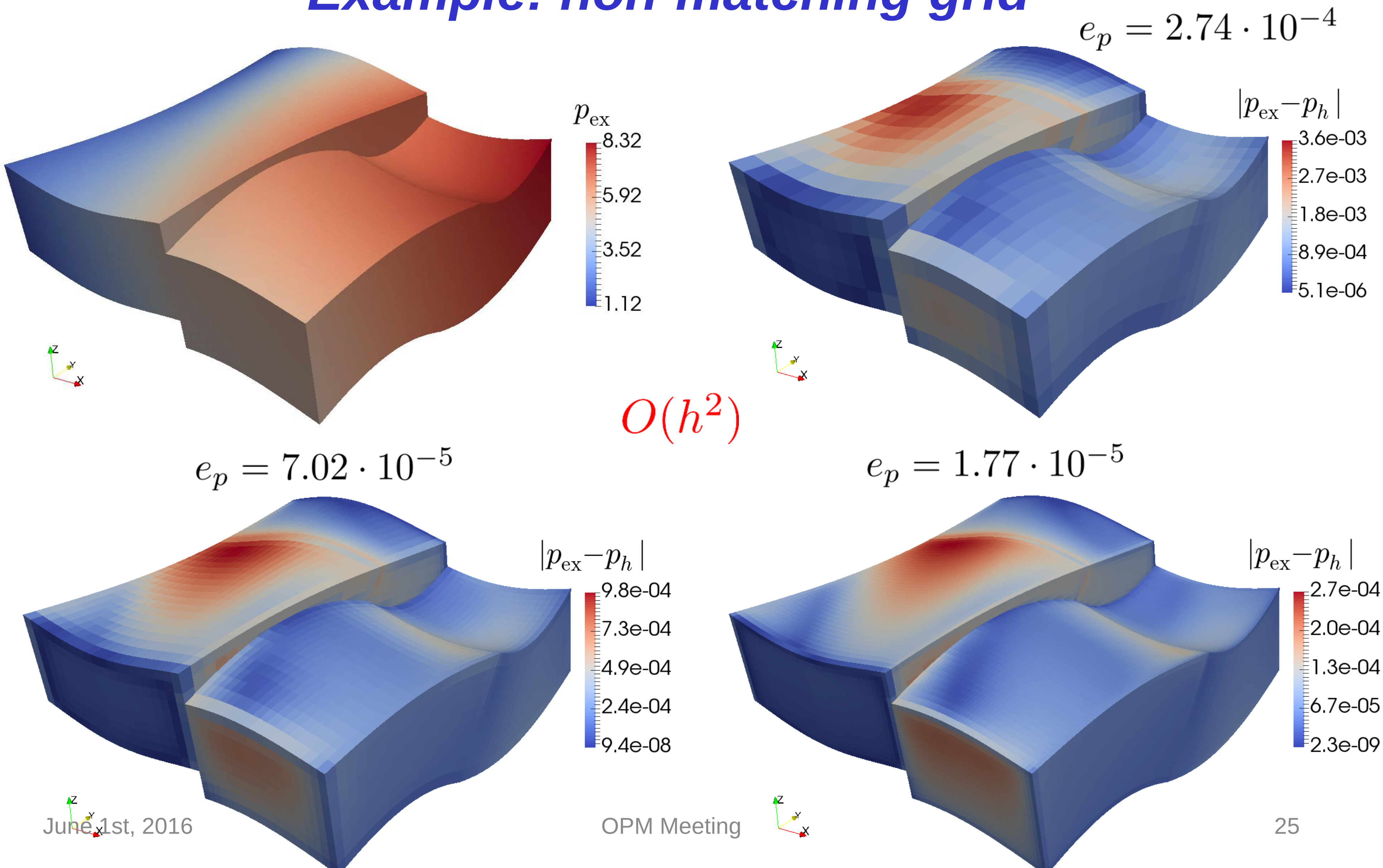
$$p_1 = 6x + 1.6y + 4xy - 2y^2 + z + 1.4$$

$$p_2 = x + 2.8y + 2xy - 2y^2 + z + 4.4$$

Example: non-matching grid

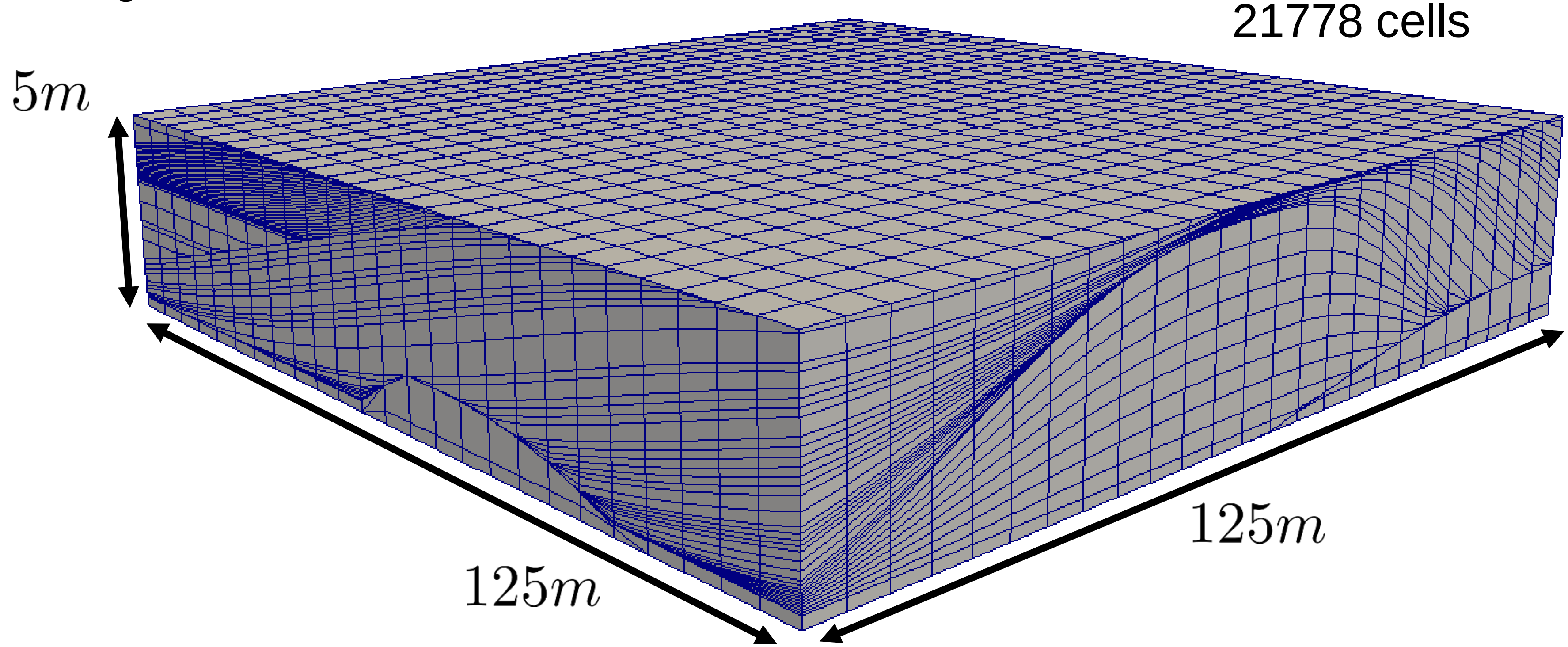


Example: non-matching grid



Example: corner-point grid

Setting:

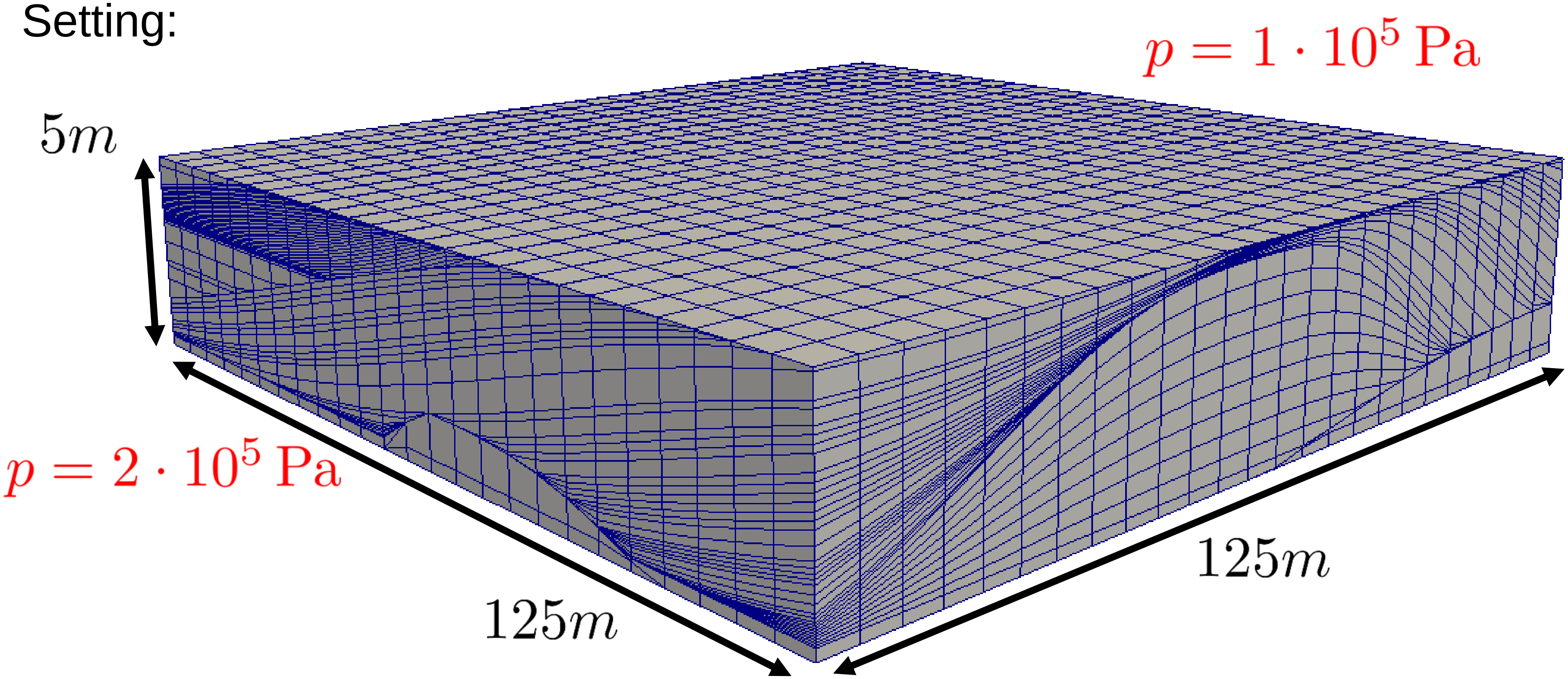


- Cell volumes differ by five orders of magnitude
- optimization: approx. 7% of cells with negative coefficients

opm-grid

Example: corner-point grid

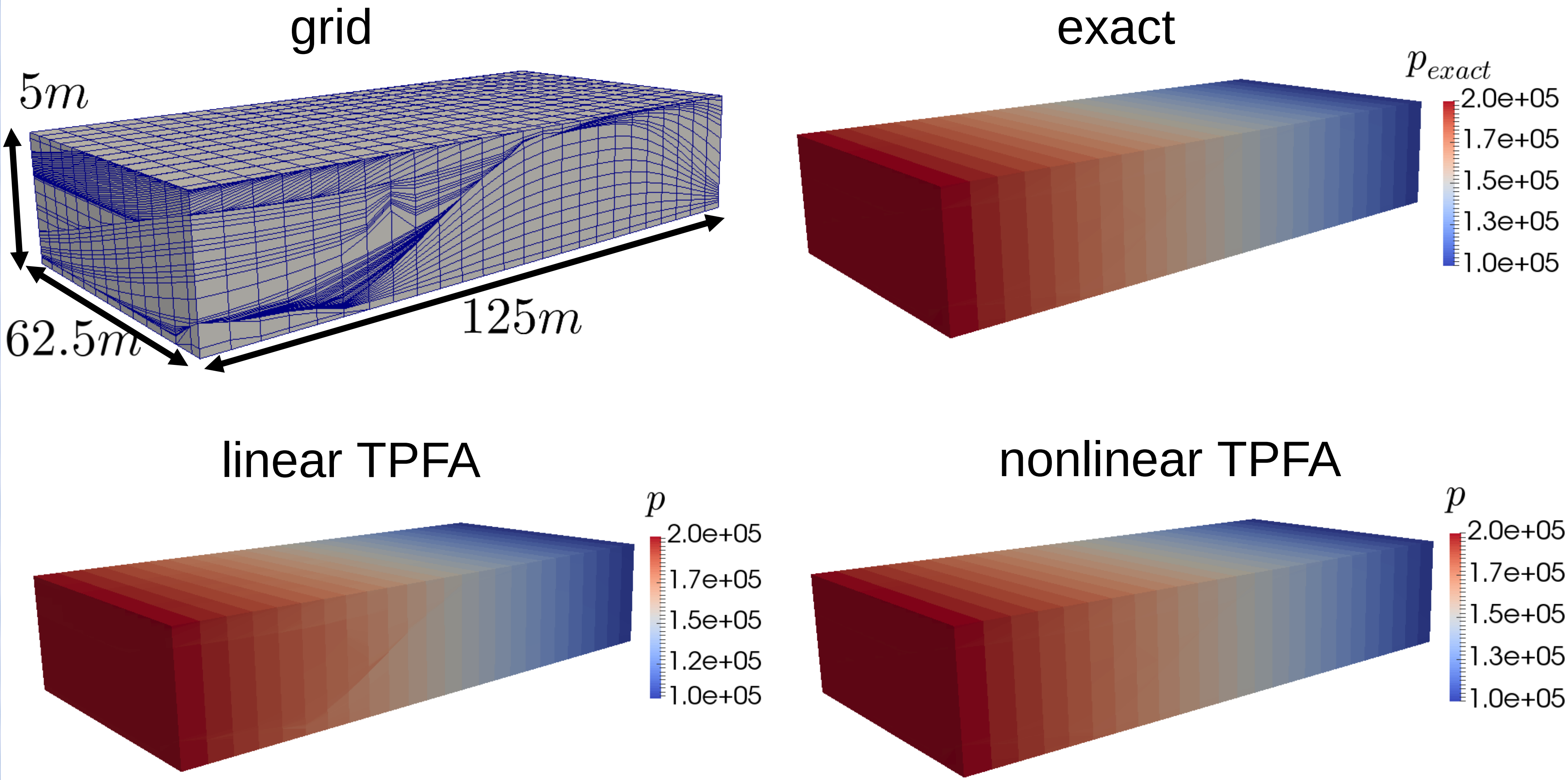
Setting:



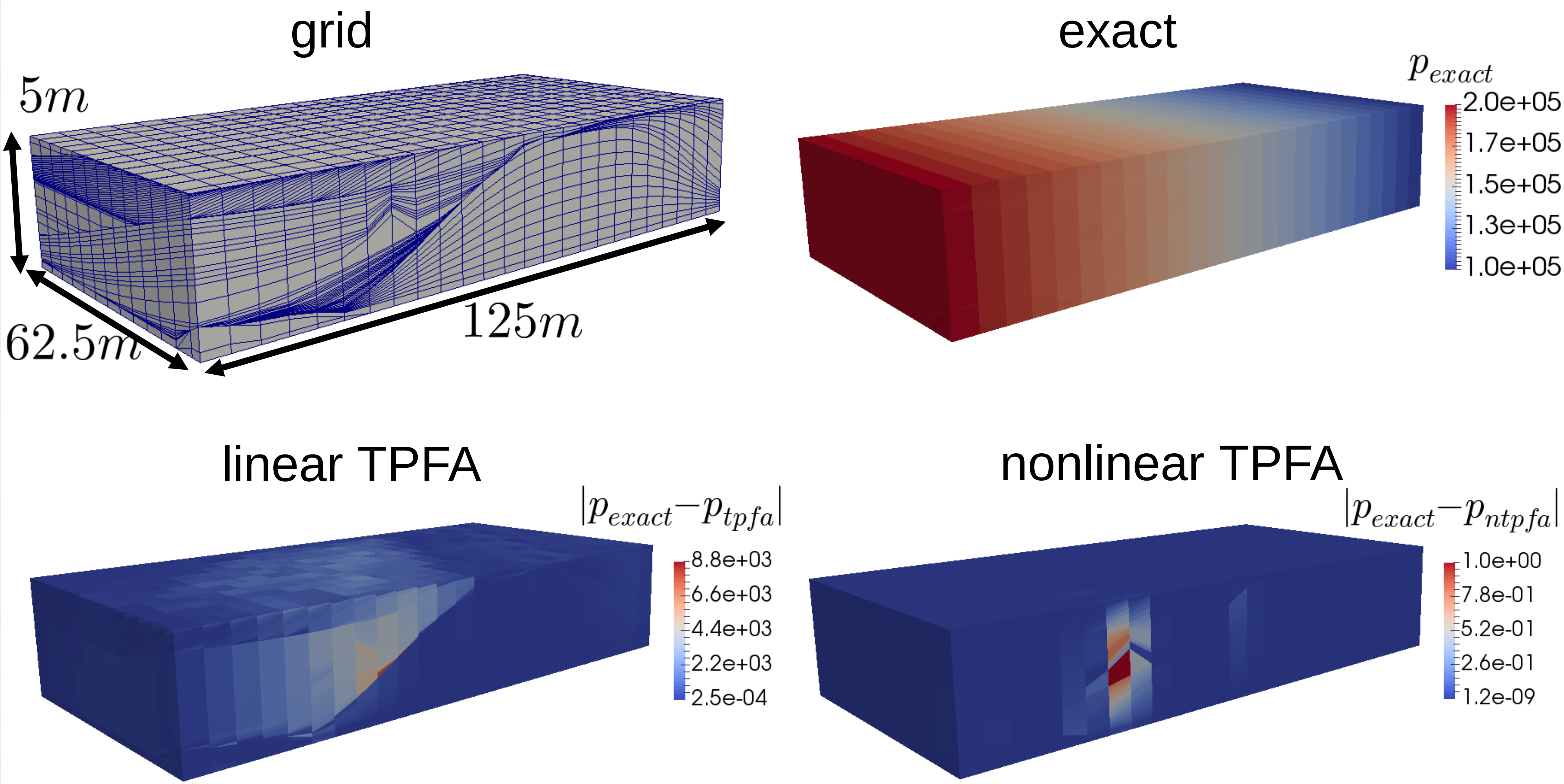
- Cell volumes differ by five orders of magnitude
- optimization: approx. 7% of cells with negative coefficients

opm-grid

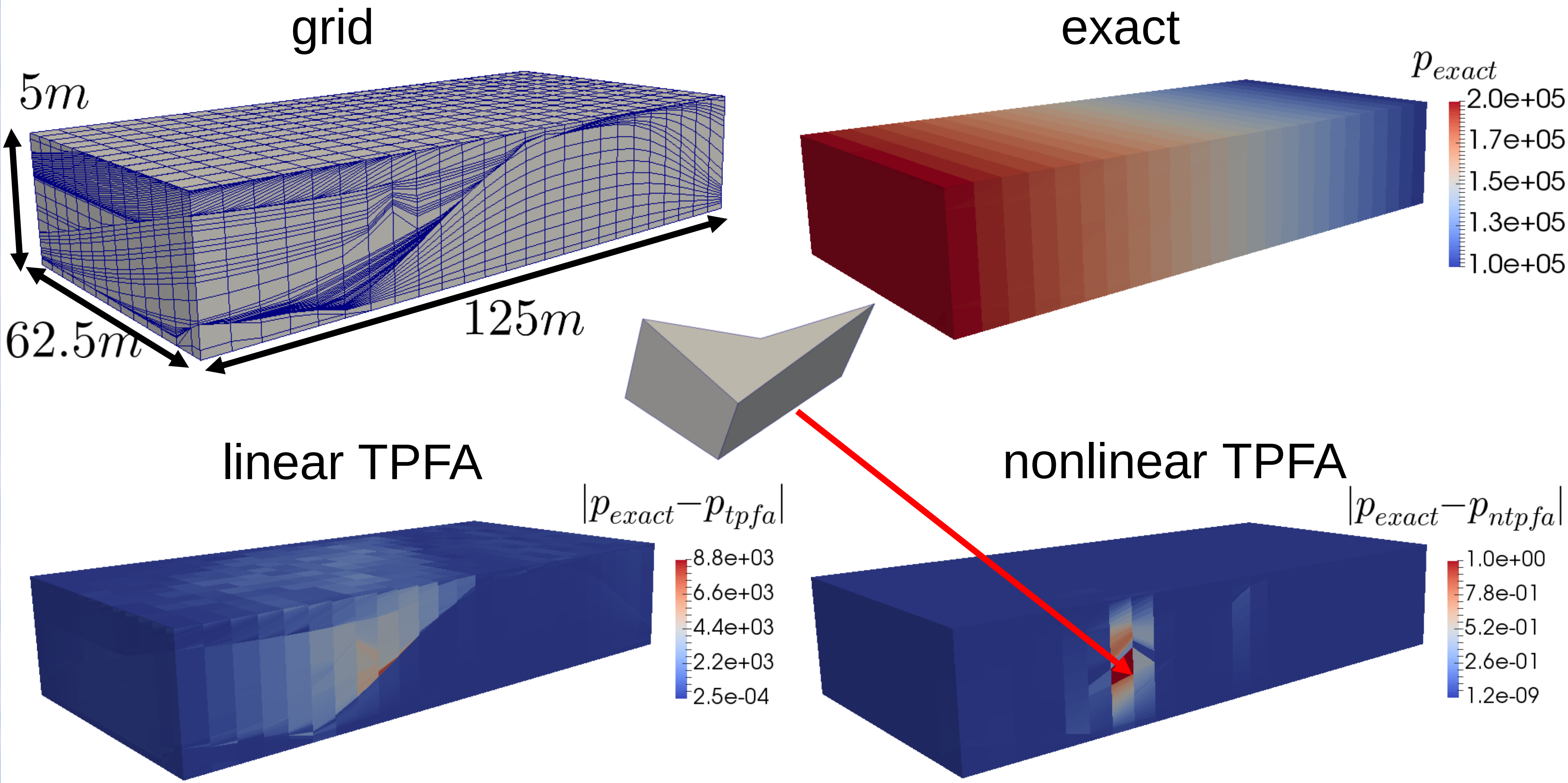
Example: corner-point grid



Example: corner-point grid

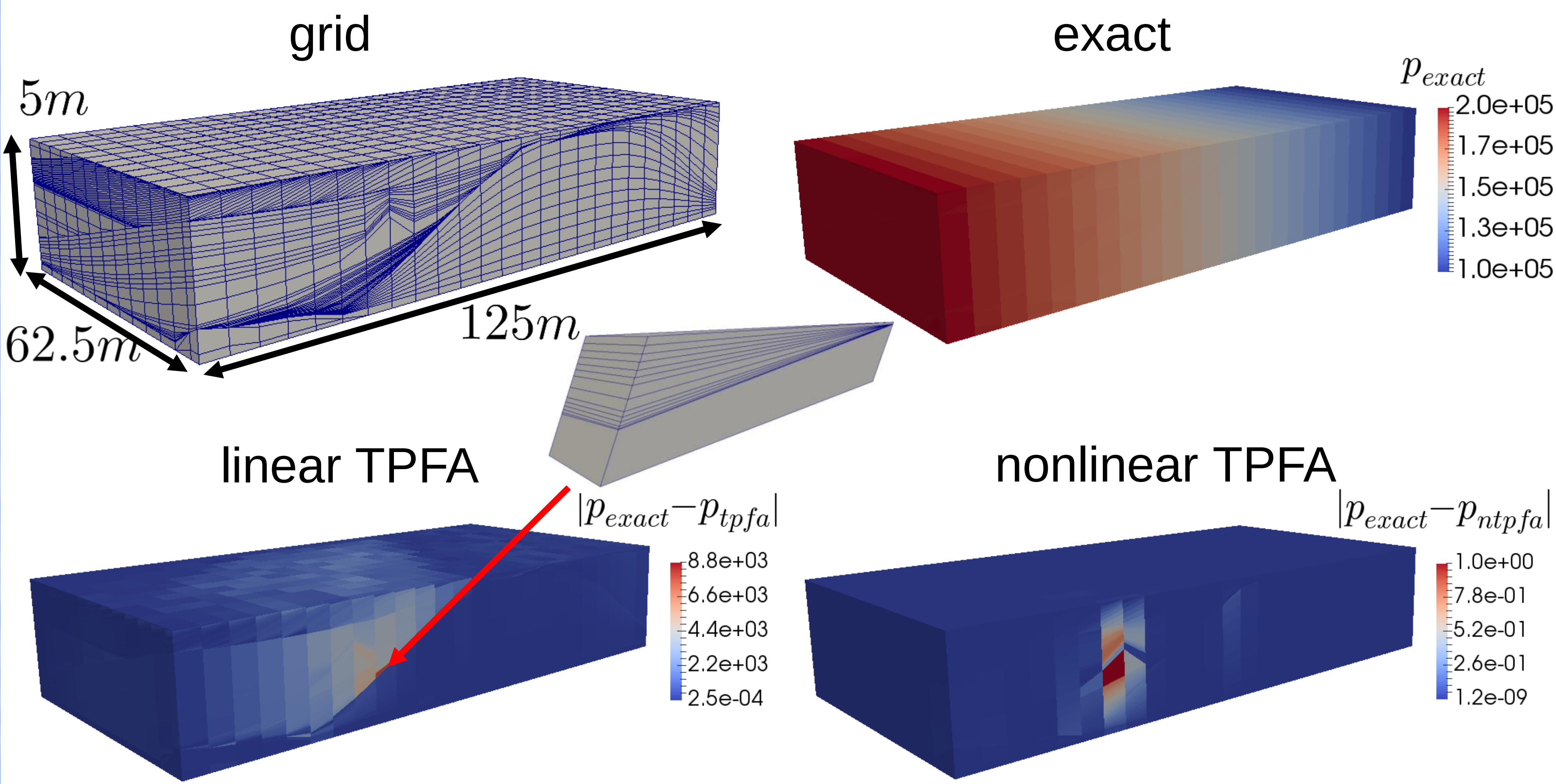


Example: corner-point grid

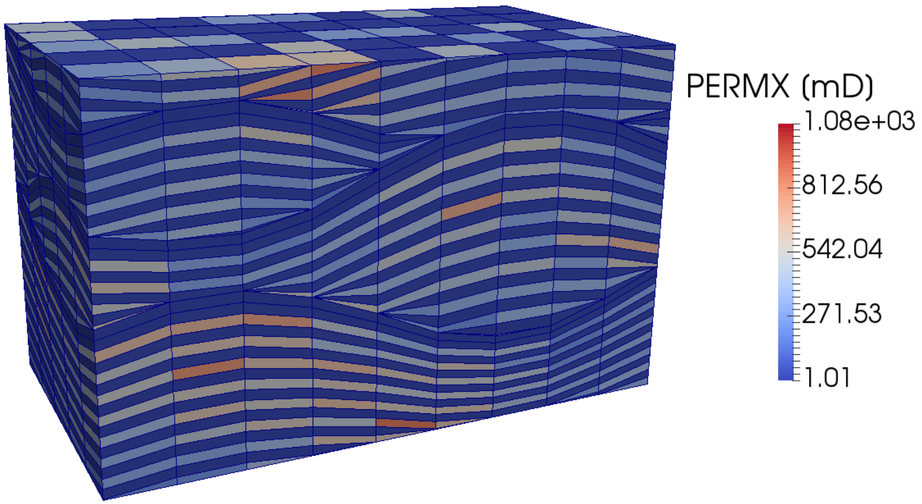


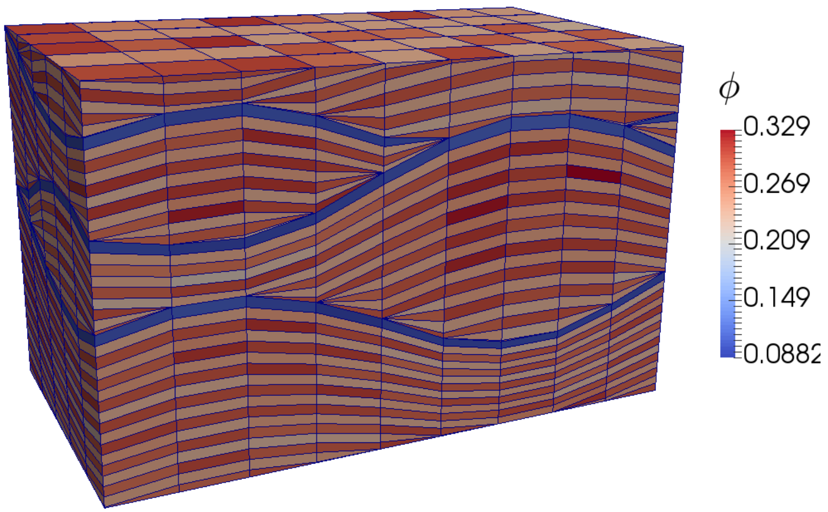
www.hydrosys.uni-stuttgart.de

Example: corner-point grid

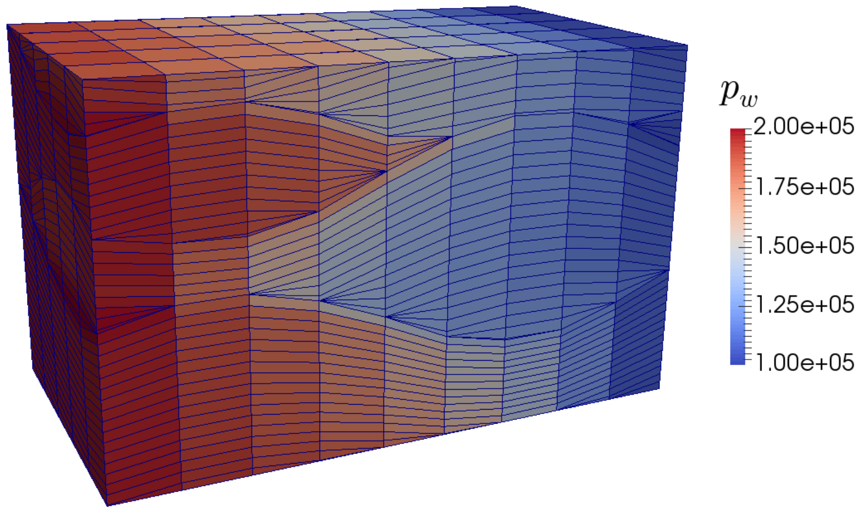


Example: corner-point grid





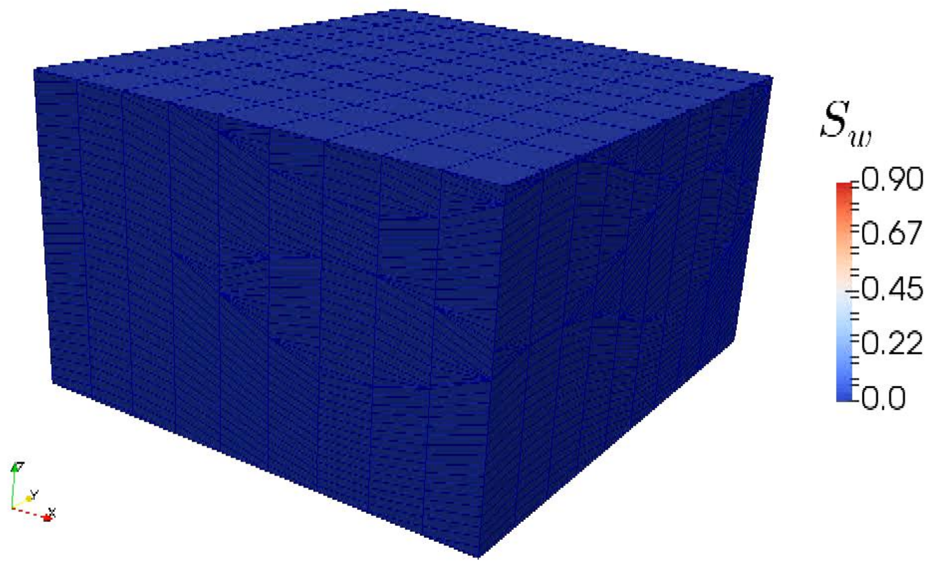
nonlinear TPFA

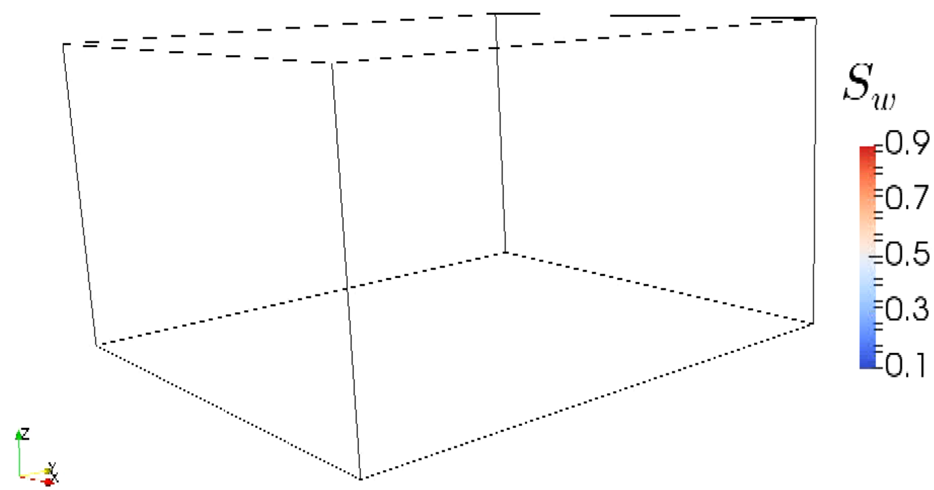


www.hydrosys.uni-stuttgart.de

Example: corner-point grid

nonlinear TPFA





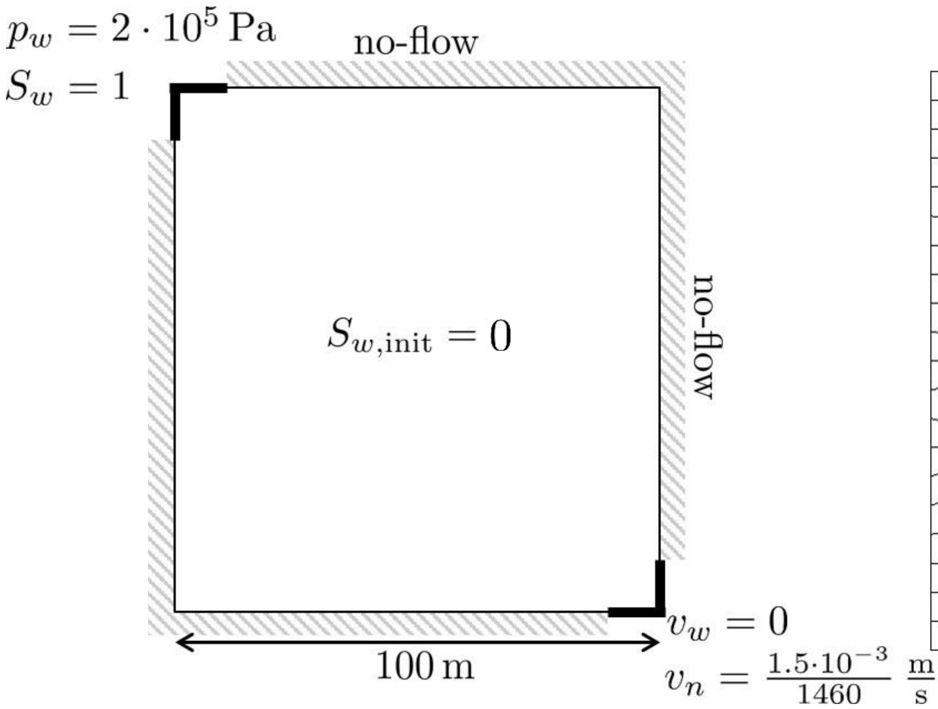
opm-grid

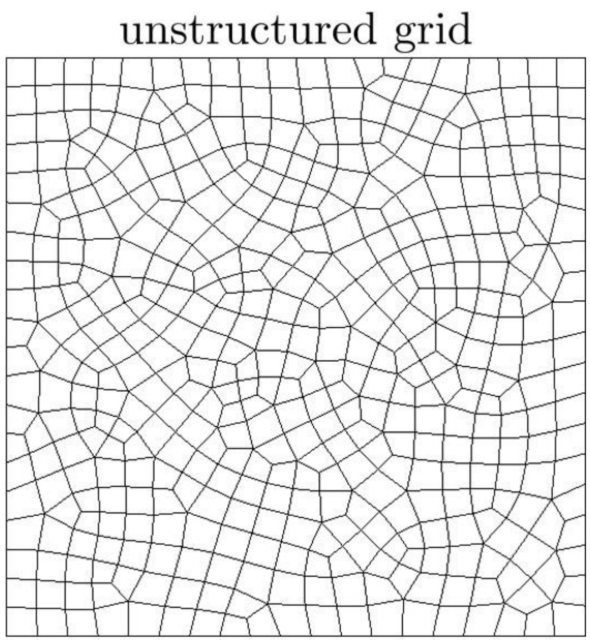
Properties of NLTPFA

- second order accuracy for pressure, first order for flux (numerical results)
- monotonicity
- handling of complex grids like corner-point grids

Comparison of different schemes

Setting:



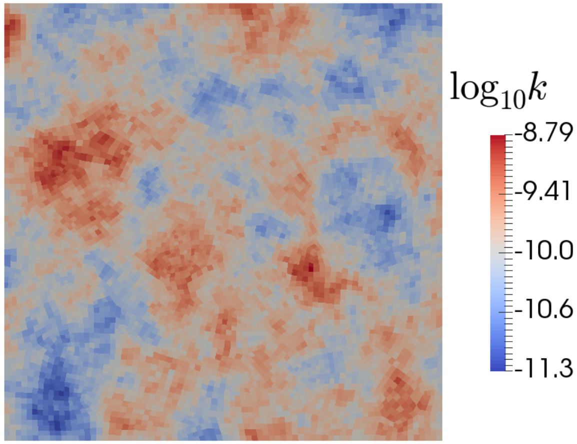


- incompressible two-phase flow
- no gravity
- no capillary pressure

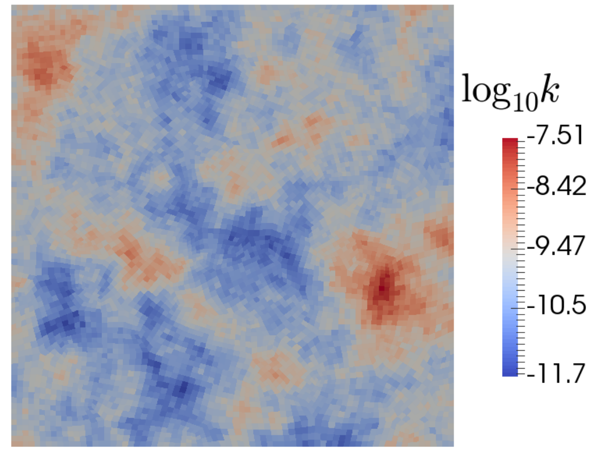
Comparison of different schemes

permeability distribution

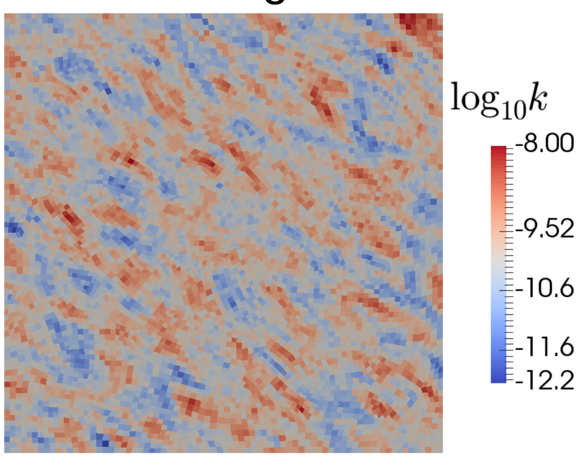
Setting 1:



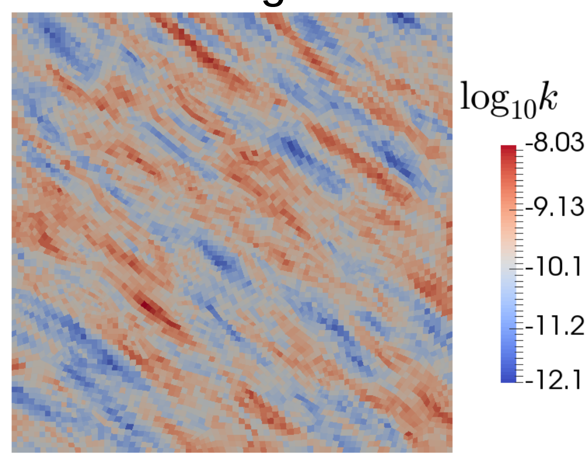
Setting 2:



Setting 3:

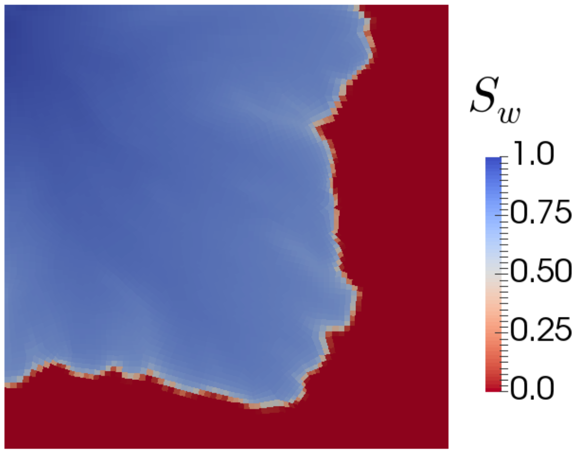


Setting 4:

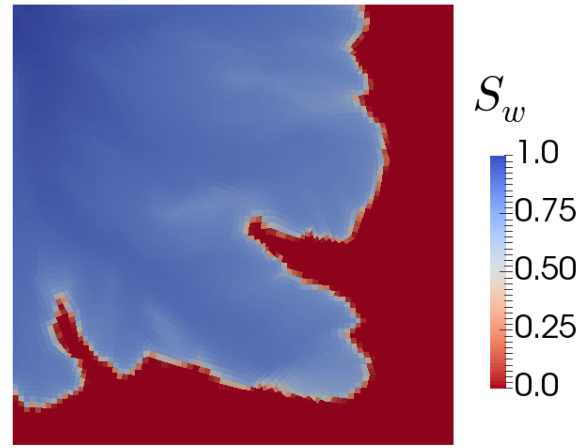


Comparison of different schemes

Setting 1:

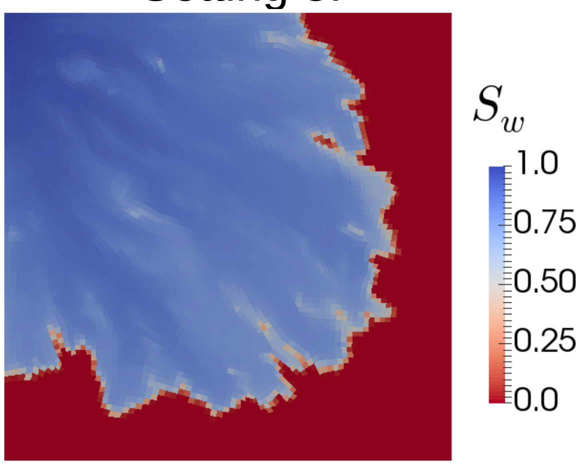


Setting 2:

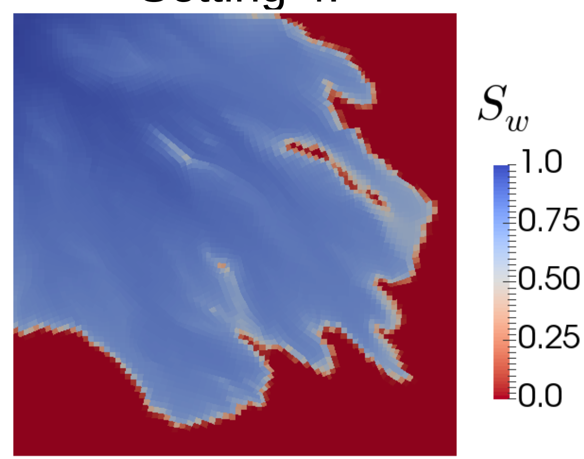


NLTPFA solution

Setting 3:



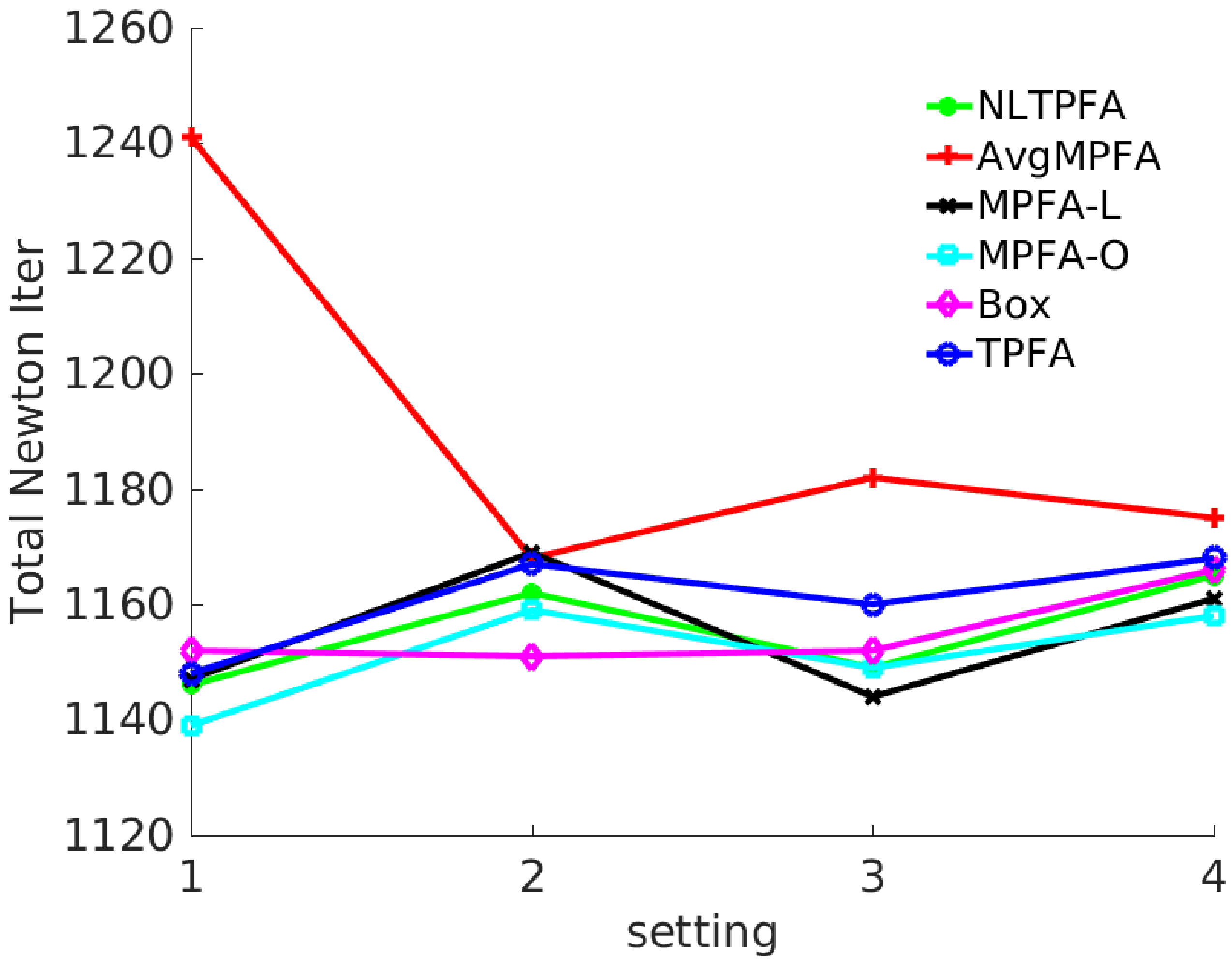
Setting 4:



similar results for all schemes:

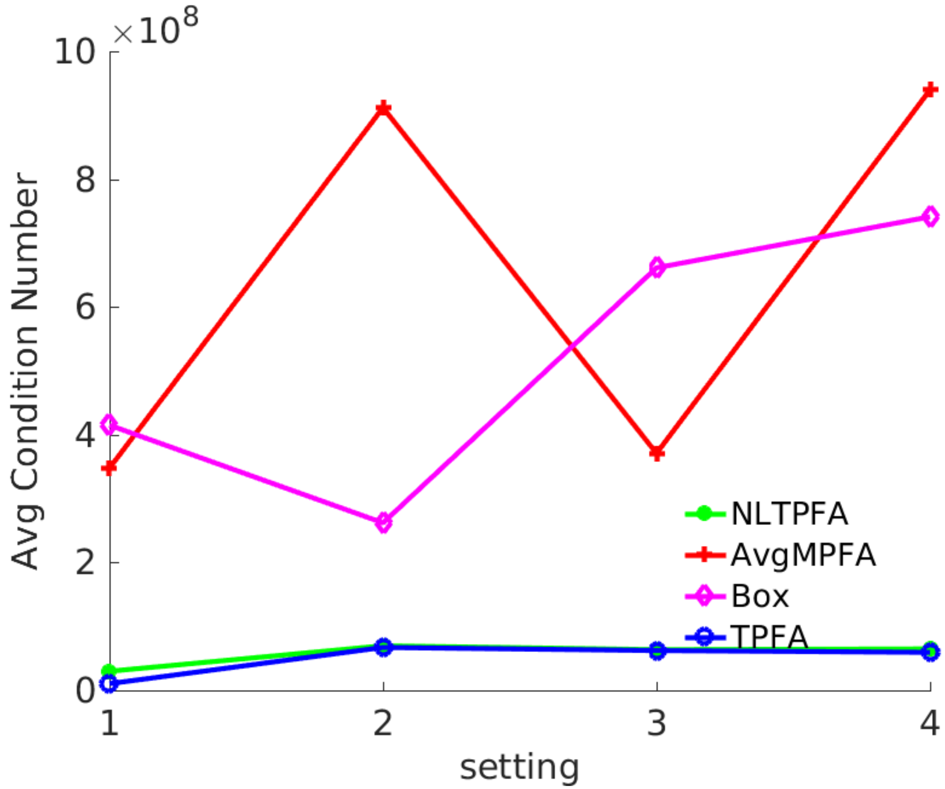
AvgMPFA
Box
TPFA
MPFA-L
MPFA-O

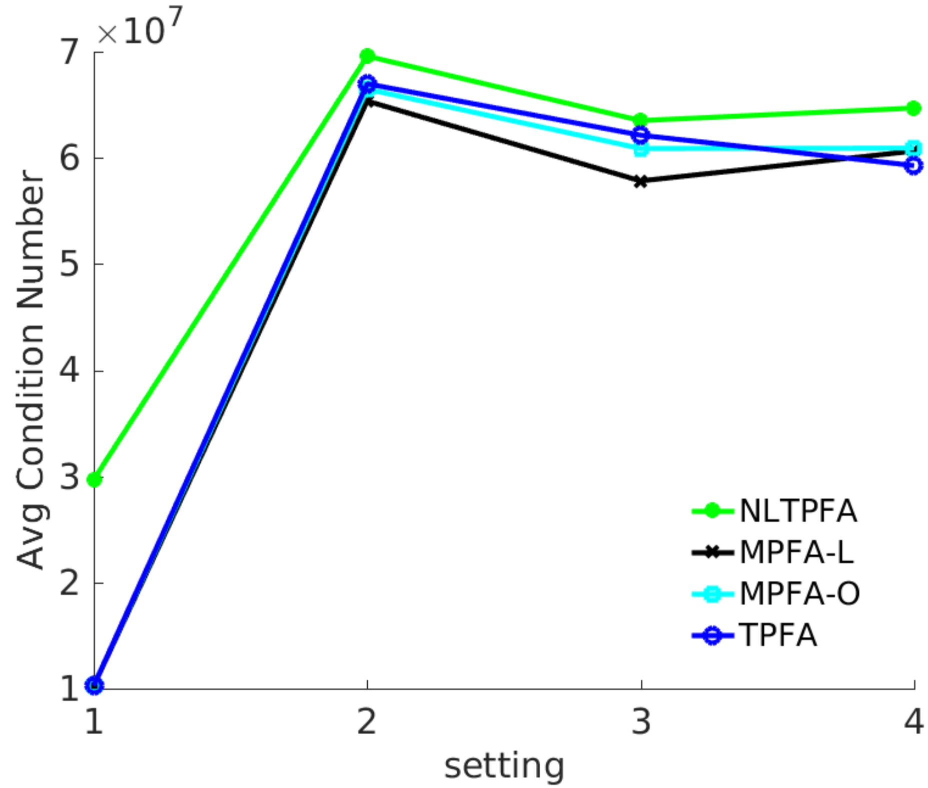
Comparison of different schemes



Comparison of different schemes

$$\overline{\kappa} = \frac{\sum_i \Delta t_i \sqrt{\kappa_1(\mathbf{J}_i)\kappa_\infty(\mathbf{J}_i)}}{\sum_i \Delta t_i}$$

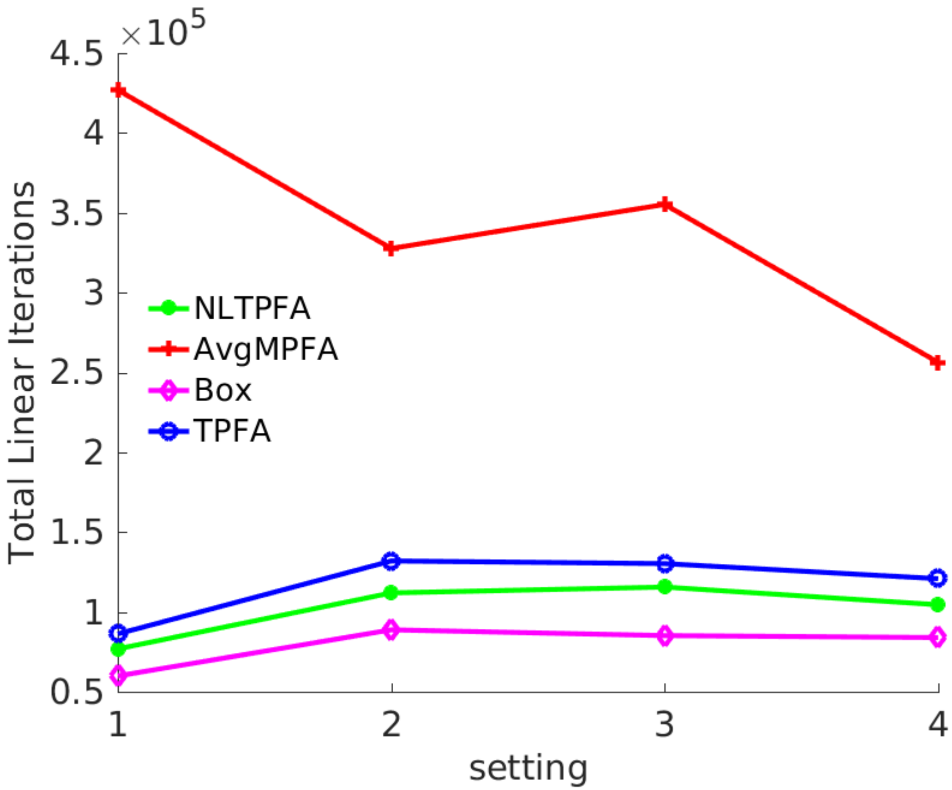


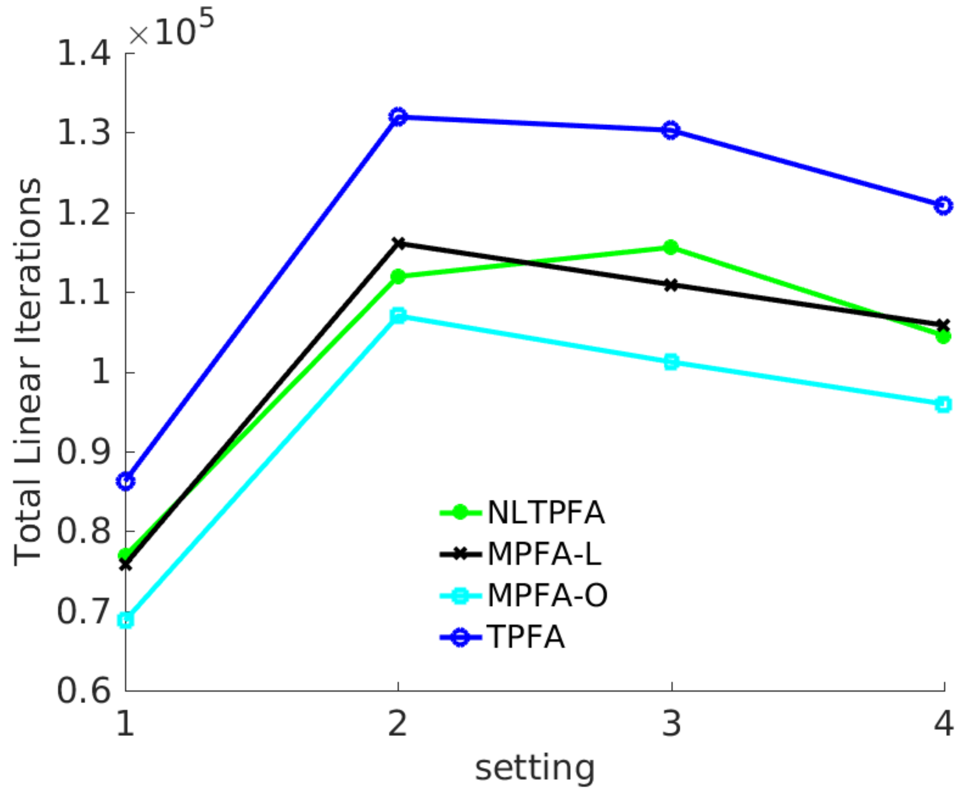


Condition number influences iterative solver behavior

Comparison of different schemes

BiCGStab solver with ILUn preconditioning





Properties of NLTPFA

- second order accuracy for pressure, first order for flux (numerical results)
- monotonicity
- handling of complex grids like corner-point grids
- NLTPFA behaves better than corresponding linear scheme (AvgMPFA)
- linear and nonlinear solvers behave similar to linear schemes

More complex example

Compressible two-phase two-component nonisothermal (2p2cni) flow equations:

Mass balance:

$$\phi \frac{\partial(\sum_\alpha \varrho_{\mathrm{mol},\alpha} x_\alpha^\kappa S_\alpha)}{\partial t} - \sum_\alpha \mathrm{div}\left\{ \frac{k_{\mathrm{r}\alpha}}{\mu_\alpha} \varrho_{\mathrm{mol},\alpha} x_\alpha^\kappa \mathbf{K}(\nabla p_\alpha - \varrho_\alpha \boldsymbol{g}) \right\}$$

$$- \sum_\alpha \mathrm{div}\left\{ \tau \phi S_\alpha \varrho_{\mathrm{mol},\alpha} \mathbf{D}_\alpha^\kappa \nabla x_\alpha^\kappa \right\} - q^\kappa = 0, \qquad \kappa \in \{\mathrm{CO}_2, \mathrm{Brine}\}.$$

Energy balance:

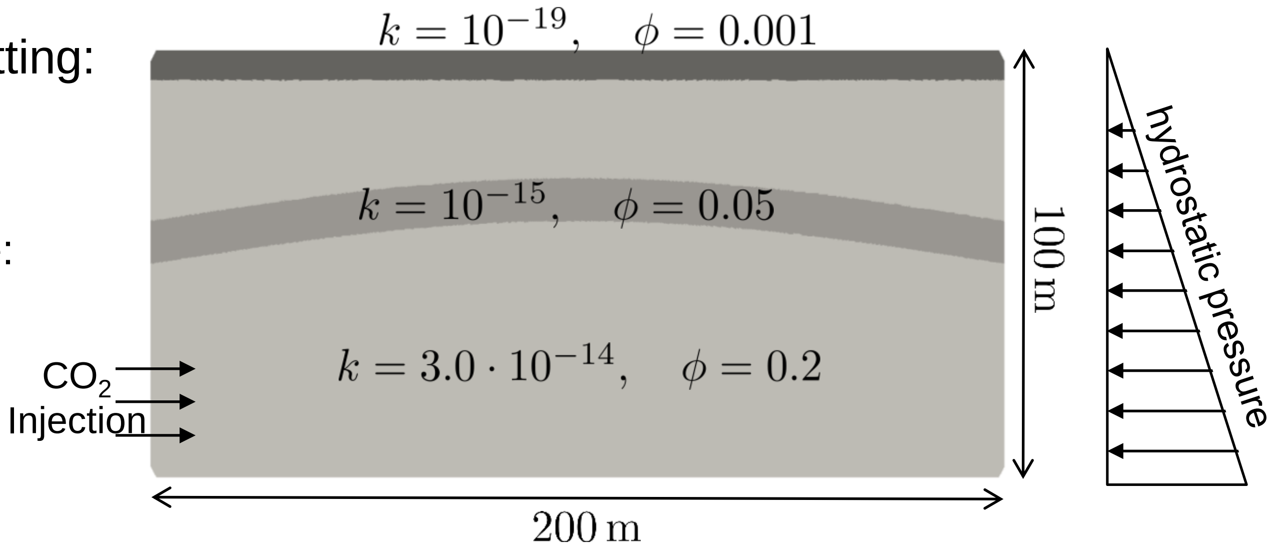
$$\phi \frac{\partial\left(\sum_\alpha \varrho_\alpha u_\alpha S_\alpha\right)}{\partial t} + (1-\phi)\frac{\partial \varrho_{\mathrm{s}} c_{\mathrm{s}} T}{\partial t} - \mathrm{div}\left(\lambda_{\mathrm{pm}} \nabla T\right)$$

$$- \sum_\alpha \mathrm{div}\left\{ \frac{k_{\mathrm{r}\alpha}}{\mu_\alpha} \varrho_\alpha h_\alpha \mathbf{K}\left(\nabla p_\alpha - \varrho_\alpha \boldsymbol{g}\right) \right\} - q^h = 0.$$

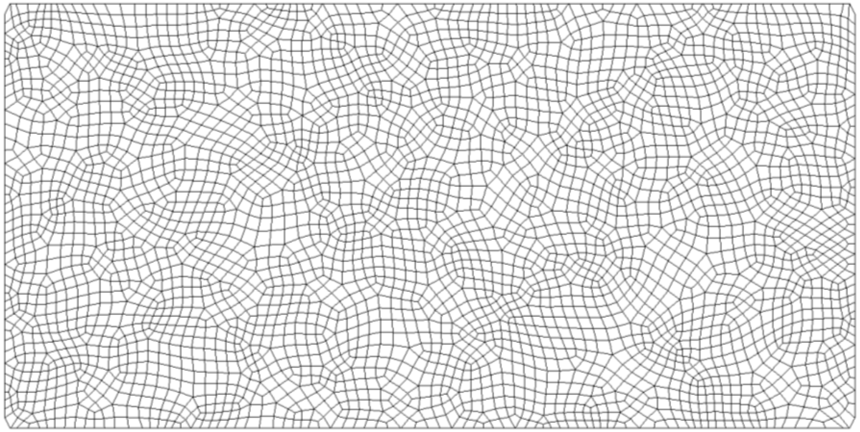
Numerical Examples

Setting:

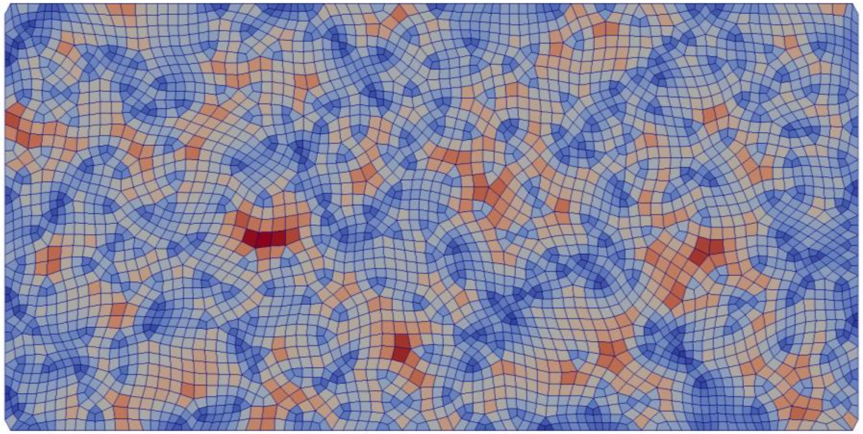
Simulation Time:
~6.3 years



Unstructured Grid:

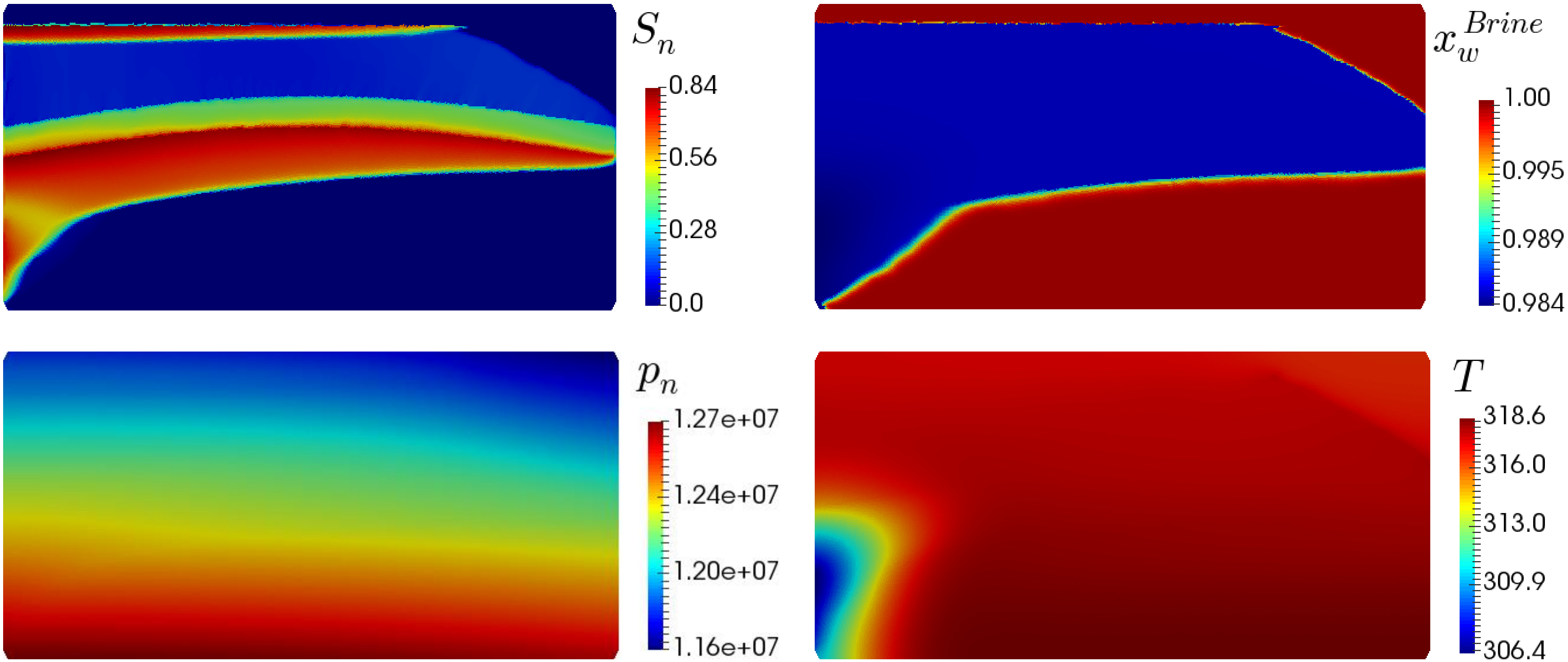


Cell Areas:



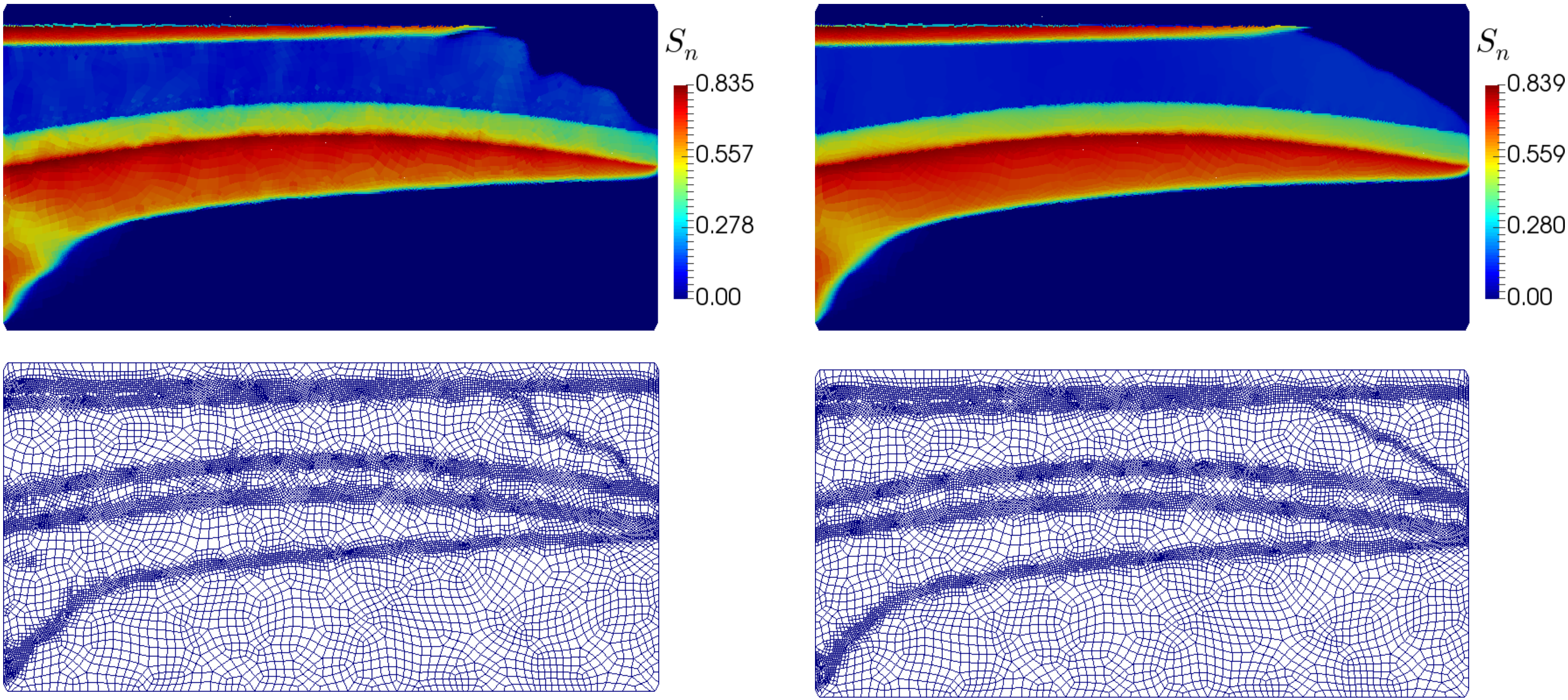
Solution of NLTPFA

After grid refinement, ~60000 cells



Adaptive Grid

TPFA: NLTPFA:



Properties of NLTPFA

- second order accuracy for pressure, first order for flux (numerical results)
- monotonicity
- handling of complex grids like corner-point grids
- NLTPFA behaves better than corresponding linear scheme (AvgMPFA)
- linear and nonlinear solvers behave similar to linear schemes
- (straight forward) applicability for physically complex nonlinear equations (2p2cni)

Conclusion

- second order accuracy for pressure, first order for flux (numerical results)
- monotonicity
- handling of complex grids like corner-point grids
- NLTPFA behaves better than corresponding linear scheme (AvgMPFA)
- linear and nonlinear solvers behave similar to linear schemes
- (straight forward) applicability for physically complex nonlinear equations (2p2cni)

Conclusion

- second order accuracy for pressure, first order for flux (numerical results)
- monotonicity
- handling of complex grids like corner-point grids
- NLTPFA behaves better than corresponding linear scheme (AvgMPFA)
- linear and nonlinear solvers behave similar to linear schemes
- (straight forward) applicability for physically complex nonlinear equations (2p2cni)

Implementation in DuMu^X

dumux-stable:

Box and TPFA method for fully-implicit porous media flow

current development:

- Unification of finite volume schemes for fully-implicit models (linear and nonlinear schemes)
- Schemes differ in face stencil classes, which provide iterators for flux calculation or matrix assembly
- Generalization of fvGeometry, fluxVars, … classes, to be able to handle grids like corner-point grids (dynamic implementation)
- Models independent of discretization

Thank you very much!

DuMu^x

References:

- Schneider, M., Flemisch, B., Helmig, R. (2016). Monotone nonlinear finite-volume method for nonisothermal two-phase two-component flow in porous media. *International Journal for Numerical Methods in Fluids. (submitted)*
- Terekhov, K., Mallison, B., Tchelepi, H. (2016). Cell-Centered Nonlinear Finite Volume Methods Preserving Positivity and the Discrete Maximum Principle for the Heterogeneous Anisotropic Diffusion Problem and Mesh Locking Effects. *(submitted)*
- Danilov, A. A., & Vassilevski, Y. V. (2009). A monotone nonlinear finite volume method for diffusion equations on conformal polyhedral meshes. *Russian Journal of Numerical Analysis and Mathematical Modelling*
- Nikitin, K., Terekhov, K., & Vassilevski, Y. (2013). A monotone nonlinear finite volume method for diffusion equations and multiphase flows. *Computational Geosciences*
- Le Potier, C. (2009). A nonlinear finite volume scheme satisfying maximum and minimum principles for diffusion operators.
- Sheng, Z., & Yuan, G. (2011). The finite volume scheme preserving extremum principle for diffusion equations on polygonal meshes. *Journal of Computational Physics*
- Nordbotten, J. M., Aavatsmark, I., & Eigestad, G. T. (2007). Monotonicity of control volume methods. *Numerische Mathematik*